MATH 240 – Practice problems for First Midterm Exam - Spring 2015

1. The determinant of the matrix $\begin{bmatrix} 2 & 0 & 1 & 0 \\ 2 & 3 & 3 & 1 \\ -3 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ is -6. What is the determinant of the matrix

$\begin{bmatrix} -3 & 2 & 1 & 2 \\ 4 & 6 & 6 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$?

of the matrix $\begin{bmatrix} 4 & 3 & 4 & 1 \\ 2 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ -3 & 2 & 1 & 2 \end{bmatrix}$?

of the matrix $\begin{bmatrix} -2 & -3 & -3 & -1 \\ 3 & -2 & -1 & -2 \\ 0 & -1 & -2 & -1 \\ -2 & 0 & -1 & 0 \end{bmatrix}$?

of the matrix $\begin{bmatrix} 4 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 3 & 3 & 1 \\ -3 & 2 & 1 & 2 \end{bmatrix}$?

The matrix $\begin{bmatrix} -3 & 2 & 1 & 2 \\ 4 & 6 & 6 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ is obtained from the original matrix by swapping the first and
third rows and doubling the second row. The row swap changes the sign of the determinant and
doubling the second row multiplies the determinant by 2. So the determinant of this matrix is
$-2(-6) = 12$.

The matrix $\begin{bmatrix} 4 & 3 & 4 & 1 \\ 2 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ -3 & 2 & 1 & 2 \end{bmatrix}$ is obtained from the original matrix by swapping the last two rows
and by adding the second row to the first row. The row swap changes the sign of the determinant
and adding the second row to the first row does not change the determinant. So the determinant of
this matrix is $-(-6) = 6$.

The matrix $\begin{bmatrix} -2 & -3 & -3 & -1 \\ 3 & -2 & -1 & -2 \\ 0 & -1 & -2 & -1 \\ -2 & 0 & -1 & 0 \end{bmatrix}$ is obtained from the original matrix by multiplying the
whole matrix by -1, and by successively swapping the first row downward until it becomes the last
row (and each of the other rows has moved up one). The negation of the entire matrix involves
multiplying all four rows by -1, so the determinant is multiplied by $(-1)^4$ (i.e., it doesn't change),

and there are three swaps altogether, which multiplies the determinant by $(-1)^3$. So the determinant of this matrix is $(-1)^4(-1)^3(-6) = 6$.

The matrix $\begin{bmatrix} 4 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 3 & 3 & 1 \\ -3 & 2 & 1 & 2 \end{bmatrix}$ is obtained from the original matrix by multiplying the first row by 2 and then by successively swapping the bottom row upward until it becomes the second row. Doubling the first row multiplies the determinant by 2 and the two swaps together multiply the determinant by $(-1)^2$. So the determinant of this matrix is $(-1)^2(2)(-6) = -12$.

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2. For each of the following subsets of the vector space P_4 of polynomials of degree less than or equal to 4, say whether or not it is a (vector) subspace of P_4. If it is not a subspace, explain why not. If it is a subspace, give its dimension and a basis for the subspace.

(a) The set of polynomials in P_4 that are even functions (i.e., for which $p(-x) = p(x)$).

(b) The set of polynomials in P_4 that are odd functions (i.e., for which $p(-x) = -p(x)$).

(c) The set of polynomials in P_4 that satisfy $p(0) = 1$ and $p(1) = 2$.

(d) The set of polynomials in P_4 that satisfy $p(0) = 0$ and $p(1) = 0$.

(e) The set of polynomials in P_4 that satisfy $p(1) = 0$, $p'(1) = 0$ and $p''(1) = 0$.

(f) The set of polynomials in P_4 that satisfy $p(1) = 1$ and $p'(1) = 2$.

(a) This is a subspace, and consists of those polynomials whose non-zero terms have even degree. So it has dimension three, and a basis is $\{1, x^2, x^4\}$.

(b) This is also a subspace, and consists of those polynomials whose non-zero terms have odd degree. So it has dimension two, and a basis is $\{x, x^3\}$.

(c) This is not a subspace, since the zero polynomial is not in it.

(d) This is a subspace – if we write a polynomial in P_4 as $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$, then the condition $p(0) = 0$ is simply $a_0 = 0$. And the condition $p(1) = 0$ says $a_0 + a_1 + a_2 + a_3 + a_4 = 0$. So this subspace has dimension $5 - 2 = 3$ and a basis is $\{x - x^4, x^2 - x^4, x^3 - x^4\}$.

(e) This is a subspace. Write the polynomial $p(x)$ as in part (d), then then condition $p(1) = 0$ is $a_0 + a_1 + a_2 + a_3 + a_4 = 0$, the condition $p'(1) = 0$ is $a_1 + 2a_2 + 3a_3 + 4a_4 = 0$ and the condition $p''(1) = 0$ is $2a_2 + 6a_3 + 12a_4 = 0$. The matrix of this system of linear equations is

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 2 & 6 & 12 & 0 \end{array}\right].$$

We can go to reduced row-echelon form as follows:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 2 & 6 & 12 & 0 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{2} R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 3 & 6 & 0 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -2 & -5 & 0 \\ 0 & 1 & 0 & -3 & -8 & 0 \\ 0 & 0 & 1 & 3 & 6 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & -3 & -8 & 0 \\ 0 & 0 & 1 & 3 & 6 & 0 \end{bmatrix}$$

So a_3 and a_4 are free variables (so the dimension of the subspace is 2), and a basis is given by

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ -6 \\ 0 \\ 1 \end{bmatrix} \right\},$$

or, more properly (as polynomials):

$$\left\{ -1 + 3x - 3x^2 + x^3, \ -3 + 8x - 6x^2 + x^4 \right\}.$$

A quicker way to do this would be to realize that a polynomial for which $p(1) = p'(1) = p''(1) = 0$ must have $(x-1)^3$ as a factor. So another basis of the same subspace would be

$$\left\{ (x-1)^3, \ x(x-1)^3 \right\} = \{ 1 - 3x + 3x^2 - x^3, \ x - 3x^2 + 3x^3 - x^4 \}.$$

(f) This is not a subspace, since the zero polynomial is not in it.

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3. Consider the matrix $A(k) = \begin{bmatrix} 1 & 1 & -2 \\ 1 & k & 0 \\ -1 & 2 & k \end{bmatrix}$.

(a) There are two values of k for which the rank of the matrix $A(k)$ is less than three. What are they?

(b) For each of those values of k, find a basis for the nullspace of $A(k)$.

(c) For one of the values of k, it is possible to solve $A(k)\boldsymbol{x} = \boldsymbol{b}$, where

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}.$$

What is the general solution of this problem for this value of k?

(a) To "determine" the values of k for which the matrix is not invertible (i.e., does not have rank 3), we calculate the determinant as follows:

$$\det A(k) = k^2 + 0 - 4 - 0 - k - 2k = k^2 - 3k - 4 = (k-4)(k-1).$$

Since the determinant is zero when $k = 4$ and $k = -1$, and non-zero otherwise, we conclude that $A(k)$ has rank 3 provided $k \notin \{4, -1\}$ and $\operatorname{rank}(A(k)) < 3$ for $k \in \{4, -1\}$.

(b) For $k = 4$, we can row-reduce:

$$\begin{bmatrix} A(4) \mid \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 4 & 0 & 0 \\ -1 & 2 & 4 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 3 & 2 & 0 \end{bmatrix} \xrightarrow[R_2 \to \frac{1}{3} R_2]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{8}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

So the rank of $A(4)$ is 2, its nullity is 1 and a basis for its nullspace (kernel) is

$$\left\{ \left[\begin{array}{c} \frac{8}{3} \\ -\frac{2}{3} \\ 1 \end{array} \right] \right\}.$$

Likewise, for $k = -1$ we can row reduce:

$$\left[\begin{array}{c|c} A(-1) & \mathbf{0} \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \end{array} \right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 3 & -3 & 0 \end{array} \right] \xrightarrow[R_2 \to -\frac{1}{2}R_2]{R_3 \to R_3 + \frac{3}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

So the rank of $A(-1)$ is also 2, its nullity is 1 and a basis for its nullspace (kernel) is

$$\left\{ \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \right\}.$$

(c) Well, let's try $k = -1$ first since that seems easier. We just have to do the same row reduction we did before, but this time augmented by the right-hand side:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 5 \\ -1 & 2 & -1 & 5 \end{array} \right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 2 & 5 \\ 0 & 3 & -3 & 5 \end{array} \right] \xrightarrow[R_2 \to -\frac{1}{2}R_2]{R_3 \to R_3 + \frac{3}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 0 & \frac{25}{2} \end{array} \right]$$

and we can already see that this system of equations has no solution, since the third equation says $0 = \frac{25}{2}$.

So we try the $k = 4$ case:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & 4 & 0 & 5 \\ -1 & 2 & 4 & 5 \end{array} \right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 3 & 2 & 5 \\ 0 & 3 & 2 & 5 \end{array} \right] \xrightarrow[R_2 \to \frac{1}{3}R_2]{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{8}{3} & -\frac{5}{3} \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Setting the free variable $x_3 = 0$, we get the particular solution $\left[\begin{array}{c} -\frac{5}{3} \\ \frac{5}{3} \\ 0 \end{array} \right]$. Put this together with the

basis for the nullspace found above to get the general solution:

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} -\frac{5}{3} \\ \frac{5}{3} \\ 0 \end{array} \right] + s \left[\begin{array}{c} \frac{8}{3} \\ -\frac{2}{3} \\ 1 \end{array} \right]$$

4. For the matrix

$$M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

determine the dimension of the subspace of 3-by-3 matrices X for which

$$MX = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Also give a basis for this subspace.

If we write the matrix X as $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$, then it looks like we have six conditions on nine variables:

$$MX = \begin{bmatrix} 2x_1 + x_4 & 2x_2 + x_5 & 2x_3 + x_6 \\ x_1 + x_7 & x_2 + x_8 & x_3 + x_9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we do get six equations in nine variables, which we write in the order:

$$
\begin{array}{rcrcrcl}
x_1 & & & + & x_7 & & & & = & 0 \\
& x_2 & & & & + & x_8 & & = & 0 \\
& & x_3 & & & & & + & x_9 & = & 0 \\
2x_1 & & & + & x_4 & & & & = & 0 \\
& 2x_2 & & & & + & x_5 & & = & 0 \\
& & 2x_3 & & & & & + & x_6 & = & 0 \\
\end{array}
$$

Row reduce the matrix corresponding to this system by adding -2 times the first, second and third rows to the fourth, fifth and sixth rows respectively, and obtain:

$$\left[\begin{array}{ccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 \\
\end{array}\right]$$

There are three free variables (x_7, x_8 and x_9) so the dimension of the nullspace is three, and a basis is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The corresponding 3-by-3 matrices (which span the space of X's for which $MX = \mathbf{0}$) are

$$\left\{ \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

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5. Let $S = \{1, x, x^2\}$ be the standard basis for the vector space P_2 of polynomials of degree less than or equal to 2.

  (a) Show that $B = \{1 + x, 1 + x^2, x + x^2\}$ is another basis for P_2.

  (b) What is the change-of-basis matrix $P_{S \leftarrow B}$ (in other words how do you go from expressing a polynomial as $a_1(1 + x) + a_2(1 + x^2) + a_3(x + x^2)$ to expressing it as $b_1(1) + b_2(x) + b_3(x^2)$)?

  (c) What is the change-of-basis matrix $P_{B \leftarrow S}$?

  (d) What is the matrix the represents the linear mapping that sends $p(x)$ to $p'(x) + 2p(x)$ with respect to the standard basis S?

  (e) What is the matrix that represents the linear mapping in part (d) with respect to the basis B?

  (a) Suppose $a_1(1 + x) + a_2(1 + x^2) + a_3(x + x^2) = 0$. Expanding this gives us

$$(a_1 + a_2) + (a_1 + a_3)x + (a_2 + a_3)x^2 = 0,$$

which would imply $a_1 + a_2 = 0$, $a_1 + a_3 = 0$ and $a_2 + a_3 = 0$. Solve this system by row reduction:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \to -R_2, \ R_3 \to \frac{1}{2}R_3]{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The last matrix is in row echelon form and clearly has rank 3, so the only possible solution is $a_1 = a_2 = a_3 = 0$. This shows that the three polynomials are linearly independent, so they must be a basis of the three-dimensional space P_2.

  (b) We did all the computation we need in part (a), and we can see that $b_1 = a_1 + a_2$, $b_2 = a_1 + a_3$ and $b_3 = a_2 + a_3$, so the change of basis matrix $P_{S \leftarrow B}$ is given by:

$$\left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array}\right]_S = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\right]_B$$

  (c) The change of basis matrix $P_{B \leftarrow S}$ is the inverse of $P_{S \leftarrow B}$, which we compute by row-reduction:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to -R_2, \ R_3 \to \frac{1}{2}R_3]{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

We conclude that $P_{B \leftarrow S} = \left[\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$.

(d) If $p(x) = b_1 + b_2 x + b_3 x^2$ (using the standard basis), then $p'(x) + 2p(x) = (2b_1 + b_2) + (2b_2 + 2b_3)x + 2b_3 x^2$. So the matrix of this linear transformation with respect to the standard basis is

$$
\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix},
$$

since we will have

$$
\begin{bmatrix} 2b_1 + b_2 \\ 2b_2 + 2b_3 \\ 2b_3 \end{bmatrix}_S = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_S.
$$

(e) To get the matrix of the same mapping with respect to the B basis, we could calculate it directly, or we could take advantage of the previous part by starting with our polynomial expressed in the B basis as $a_1(1 + x) + a_2(1 + x^2) + a_3(x + x^2)$, or equivalently as the vector

$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_B,
$$

translate this into the standard basis by multiplying by the change of basis matrix $P_{S \leftarrow B}$, then using the matrix from part (d) which carries out the linear map using the standard basis, and then multiplying the result by the change of basis matrix $P_{B \leftarrow S}$ to express the result in terms of the B basis. Thus, the matrix of the transformation with respect to the B basis is

$$
P_{B \leftarrow S} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} P_{S \leftarrow B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 4 \\ 0 & 2 & 2 \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{5}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix}
$$

This last matrix tells us that if

$$
p(x) = a_1(1 + x) + a_2(1 + x^2) + a_3(x + x^2)
$$

then

$$
p'(x) + 2p(x) = (\frac{5}{2}a_1 + a_2 + \frac{3}{2}a_3)(1 + x) + (\frac{1}{2}a_1 + a_2 - \frac{1}{2}a_3)(1 + x^2) + (-\frac{1}{2}a_1 + a_2 + \frac{5}{2}a_3)(x + x^2),
$$

which you can check.

6. (a) Can the vector $\begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \end{bmatrix}$ be represented as a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 6 \\ 2 \end{bmatrix}$,

and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$? If not, explain why not. If so, how? (Be precise – if there is more than one way to do

it, give all possible ways).

(b) Same question, but for the vector $\begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix}$

(a) The question is asking whether we can find values x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 6 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \end{bmatrix},$$

in other words, we need to solve the system

$$
\begin{array}{rrrrrrr}
x_1 & + & x_2 & & & = & 2 \\
& & x_2 & & & = & 1 \\
2x_1 & + & 6x_2 & + & x_3 & = & 5 \\
x_1 & + & 2x_2 & + & x_3 & = & 0
\end{array}
$$

which we do by row-reduction. In fact, looking ahead to part (b), we'll solve that system at the same time:

$$
\left[\begin{array}{ccc|c|c}
1 & 1 & 0 & 2 & 2 \\
0 & 1 & 0 & 1 & 1 \\
2 & 6 & 1 & 5 & 4 \\
1 & 2 & 1 & 0 & 1
\end{array}\right]
\xrightarrow[R_4 \to R_4 - R_1]{R_3 \to R_3 - 2R_1}
\left[\begin{array}{ccc|c|c}
1 & 1 & 0 & 2 & 2 \\
0 & 1 & 0 & 1 & 1 \\
0 & 4 & 1 & 1 & 0 \\
0 & 1 & 1 & -2 & -1
\end{array}\right]
\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - 4R_2}
\left[\begin{array}{ccc|c|c}
1 & 1 & 0 & 2 & 2 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & -3 & -4 \\
0 & 0 & 1 & -3 & -2
\end{array}\right]
$$

$$
\xrightarrow{R_4 \to R_4 - R_3}
\left[\begin{array}{ccc|c|c}
1 & 1 & 0 & 2 & 2 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & -3 & -4 \\
0 & 0 & 0 & 0 & 2
\end{array}\right]
\xrightarrow{R_1 \to R_1 - R_2}
\left[\begin{array}{ccc|c|c}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & -3 & -4 \\
0 & 0 & 0 & 0 & 2
\end{array}\right]
$$

From this, we see that the reduced row echelon form for the problem in part (a) is

$$
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

which tells us that $x_1 = 1$, $x_2 = 1$ and $x_3 = -3$ is the only solution (since there are no free variables), and we can write

$$
\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 6 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \end{bmatrix}.
$$

(b) From the calculation above, the reduced row echelon form for this problem is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 2 \end{array}\right].$$

But this system has no solutions (the last equation says $0 = 2$) so we conclude that there is no way to write the vector $\begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 6 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

---

7. Let $A = \begin{bmatrix} 1 & 0 & -1 & -2 & 0 & 0 \\ -2 & -1 & 0 & 2 & 0 & -1 \end{bmatrix}$.

(a) Explain why the matrix $A^T(AA^T)^{-1}$ would be a right inverse for A, provided it exists.

(b) Calculate $A^T(AA^T)^{-1}$ and show that it is a right inverse for A. (Sorry about the fractions!)

(a) "If it exists" means "if $(AA^T)^{-1}$ exists" — assuming that it does, we can show that the given matrix is a right inverse for A by multiplying A on the right by $A^T(AA^T)^{-1}$:

$$A[A^T(AA^T)^{-1}] = (AA^T)(AA^T)^{-1} = I$$

since the product of a matrix with its inverse is the identity (the 2-by-2 identity matrix in this case).

(b) First we calculate AA^T:

$$AA^T = \begin{bmatrix} 1 & 0 & -1 & -2 & 0 & 0 \\ -2 & -1 & 0 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ -1 & 0 \\ -2 & 2 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ -6 & 10 \end{bmatrix}$$

Next we calculate $(AA^T)^{-1}$ by row reduction:

$$\left[\begin{array}{cc|cc} 6 & -6 & 1 & 0 \\ -6 & 10 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \to \frac{1}{6}R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{cc|cc} 1 & -1 & \frac{1}{6} & 0 \\ 0 & 4 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{4}R_2} \left[\begin{array}{cc|cc} 1 & -1 & \frac{1}{6} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{12} & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \end{array}\right]$$

So $(AA^T)^{-1} = \begin{bmatrix} \frac{5}{12} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$ and so a right inverse for A is

$$A^T(AA^T)^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ -1 & 0 \\ -2 & 2 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{5}{12} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \\ -\frac{5}{12} & -\frac{1}{4} \\ -\frac{1}{3} & 0 \\ 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

(You can check that this is correct by multiplying A on the right by it and obtaining the 2-by-2 identity matrix.)

Extra problems

1. Find *all* solutions of the following equations — or show that there is none.

(a)
$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & x_3 & - & 2x_4 & = & 0 \\ x_1 & + & x_2 & + & 3x_3 & - & 2x_4 & = & 0 \end{array}$$

(b)
$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 1 \\ x_1 & - & x_2 & = & 3 \\ 2x_1 & + & x_2 & = & 3 \end{array}$$

(a) These are homogeneous equations, so the set of solutions is a subspace of $\mathbb{R}^4$. To find a basis for it, we start with the augmented matrix of the system and row-reduce:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 3 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

There are two free variables, x_2 and x_4. Setting each equal to 1 and the other 0 in turn, we get the following basis for the set of solutions:

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(b) These are inhomogeneous equations, and more equations than unknowns, so there may be no solutions. But we start again with the augmented matrix of the system and row-reduce:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 2 & 1 & 3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_2]{R_2 \to -\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

The system is consistent, since the last row just says $0 = 0$, and there are no free variables. Therefore the unique solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

---

2. Determine whether the matrix

$$\begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

is invertible. Find its inverse if it is.

The determinant of the matrix is

$$30 + 30 + 21 - 27 - 25 - 28 = 1$$

Which is not zero, so the matrix is invertible (and we don't expect fractions in the inverse!). We find the inverse by row reduction:

$$
\left[\begin{array}{ccc|ccc}
3 & 5 & 7 & 1 & 0 & 0 \\
1 & 2 & 3 & 0 & 1 & 0 \\
2 & 3 & 5 & 0 & 0 & 1
\end{array} \right]
\xrightarrow[R_2 \to R_2 - 3R_1, \ R_3 \to R_3 - 2R_1]{R_1 \leftrightarrow R_2}
\left[\begin{array}{ccc|ccc}
1 & 2 & 3 & 0 & 1 & 0 \\
0 & -1 & -2 & 1 & -3 & 0 \\
0 & -1 & -1 & 0 & -2 & 1
\end{array} \right]
$$

$$
\xrightarrow[R_2 \to -R_2]{R_3 \to R_3 - R_2}
\left[\begin{array}{ccc|ccc}
1 & 2 & 3 & 0 & 1 & 0 \\
0 & 1 & 2 & -1 & 3 & 0 \\
0 & 0 & 1 & -1 & 1 & 1
\end{array} \right]
\xrightarrow[R_1 \to R_1 - 3R_3]{R_2 \to R_2 - 2R_3}
\left[\begin{array}{ccc|ccc}
1 & 2 & 0 & 3 & -2 & -3 \\
0 & 1 & 0 & 1 & 1 & -2 \\
0 & 0 & 1 & -1 & 1 & 1
\end{array} \right]
$$

$$
\xrightarrow{R_1 \to R_1 - 2R_2}
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 1 & -4 & 1 \\
0 & 1 & 0 & 1 & 1 & -2 \\
0 & 0 & 1 & -1 & 1 & 1
\end{array} \right]
$$

So the inverse of the matrix is $\begin{bmatrix} 1 & -4 & 1 \\ 1 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$.

---

3. Let A and B be 2-by-2 matrices. We say that A and B *commute*, if $AB = BA$. Show that if A and B both commute with

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

then A commutes with B also.

We begin by characterizing the matrices that commute with $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is such a matrix, then $AJ - JA = \mathbf{0}$, that is to say

$$
\begin{aligned}
AJ - JA &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\
&= \begin{bmatrix} -a_{12} & a_{11} \\ -a_{22} & a_{21} \end{bmatrix} - \begin{bmatrix} a_{21} & a_{22} \\ -a_{11} & -a_{12} \end{bmatrix} \\
&= \begin{bmatrix} -a_{12} - a_{21} & a_{11} - a_{22} \\ a_{11} - a_{22} & a_{21} + a_{12} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

This means that A commutes with J if and only if $a_{21} = -a_{12}$ and $a_{11} = a_{22}$, so if A and B both commute with J then they must be of the form:

$$A = \begin{bmatrix} p & q \\ -q & p \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} r & s \\ -s & r \end{bmatrix}.$$

And then we can check that A and B commute with each other

$$AB - BA = \begin{bmatrix} p & q \\ -q & p \end{bmatrix} \begin{bmatrix} r & s \\ -s & r \end{bmatrix} - \begin{bmatrix} r & s \\ -s & r \end{bmatrix} \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$$

$$= \begin{bmatrix} pr - qs & ps + qr \\ -ps - qr & pr - qs \end{bmatrix} - \begin{bmatrix} pr - qs & ps + qr \\ -ps - qr & pr - qs \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(You might notice the similarity between the multiplication of A and B and the multiplication of the complex numbers $p + iq$ and $r + is$).

---

4. Use Gaussian elimination to find the values of b for which the following linear system has a solution. Find the corresponding solution(s).

$$\begin{aligned} x_2 &+ 2x_3 &= 4 \\ x_1 + 2x_2 &+ 5x_3 &= 6 \\ - x_2 &- 2x_3 &= b \end{aligned}$$

We start with the augmented matrix of the system and row-reduce:

$$\begin{bmatrix} 0 & 1 & 2 & | & 4 \\ 1 & 2 & 5 & | & 6 \\ 0 & -1 & -2 & | & b \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 5 & | & 6 \\ 0 & 1 & 2 & | & 4 \\ 0 & -1 & -2 & | & b \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 5 & | & 6 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & 0 & | & 4+b \end{bmatrix}$$

Since the third equation now reads $0 = 4 + b$, there is no solution unless $b = -4$. In the case that $b = -4$, we continue the row reduction as follows:

$$\begin{bmatrix} 1 & 2 & 5 & | & 6 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1 & | & -2 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This shows that x_3 is the only free variable, a particular solution of the system is $\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$ (found by setting $x_3 = 0$, and a basis for the solution of the associated homogeneous system (found by setting $x_3 = 1$ and the right side equal to zero) is $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$. The general solution of the problem in this case is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 - s \\ 4 - 2s \\ s \end{bmatrix}.$$

---

5. Compute the determinant of

$$\begin{bmatrix} 1 & 6 & 11 & 16 & 21 \\ 2 & 7 & 12 & 17 & 22 \\ 3 & 8 & 13 & 18 & 23 \\ 4 & 9 & 14 & 19 & 24 \\ 5 & 10 & 15 & 20 & 25 \end{bmatrix}$$

Hint: Very little computing is needed.

If you subtract the first row from the second row, then the second row will become $[1, 1, 1, 1, 1]$ (and the determinant does not change). Then subtract the third row from the fourth row and the fourth row will be $[1, 1, 1, 1, 1]$ and the determinant is still the same as that of the original matrix. But the determinant of a matrix with two identical rows is zero, so the determinant of the original matrix is zero.

True/False questions:

1. Let A be a square matrix. If $A^3 = \mathbf{0}$ then $\det(A) = 0$.

True. Since $\det(A^3) = (\det(A))^3$, we have that the numbers $(\det(A))^3 = 0$, and so $\det(A) = 0$.

2. Let A, B and C be n-by-n matrices. If $AB = AC$ and A is invertible, then $B = C$.

True. Multiply the given equation by A^{-1} on the left. So $A^{-1}AB = A^{-1}AC$, i.e., $IB = IC$, i.e, $B = C$.

3. Let L be an invertible map from the plane $\mathbb{R}^2$ to itself that has the property that it is its own inverse. Then $L = \pm I$, where I is the identity map.

False. L could be the map represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which is reflection across the x-axis.

4. If A and B are n-by-n matrices with A invertible, then $(ABA^{-1})^2 = AB^2A^{-1}$.

True. A (careful!) computation: $(ABA^{-1})^2 = (ABA^{-1})(ABA^{-1}) = AB(A^{-1}A)BA^{-1} = ABIBA^{-1} = AB^2A^{-1}$.

5. Say A is a 4-by-4 matrix for which $\det(A) = -3$. Then $\det(2A) = -6$.

False. The matrix $2A$ is obtained from A by multiplying each of its four rows by 2, and each time a row is multiplied by 2 the determinant is multiplied by 2. So $\det(2A) = 2^4 \det(A) = 2^4(-6) = -96$ in this case.

6. Let A, B be two real 5-by-5 matrices. Then $\det(A + B) = \det(A) + \det(B)$.

False. Choose A and B both to be the identity matrix. Then $\det(A) = \det(B) = 1$ but $\det(A + B) = 32$.

7. If the 3-by-3 matrices A and B are both nonsingular, then $A + B$ is also nonsingular.

False. Choose $A = I$ and $B = -I$, for instance.

8. If the 4-by-4 matrices A and B are both symmetric, then $A + B$ is also symmetric.

True. If $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$ then clearly $a_{ij} + b_{ij} = a_{ji} + b_{ji}$.

9. Let a, b, c and d be *non-zero* vectors in $\mathbb{R}^3$. It is impossible that a, b, c and d are linearly independent.

True. Since $\mathbb{R}^3$ is three-dimensional, the maximum number of linearly independent vectors is three.

For the following three questions, consider a system of linear algebraic equations written in matrix form

$$A\boldsymbol{x} = \boldsymbol{b},$$

where A is an n-by-n matrix with $\det(A) = 0$,

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$x_1, \ldots, x_n$ are the unknowns.

10. It is impossible that for some vector $\boldsymbol{b}$ there is exactly one solution.

True. Since $\det(A) = 0$, the rank of A is less than n, so there must always be at least one free variable (i.e., the nullspace of A must be at least one-dimensional, and you can add any non-zero member of the nullspace of A to a particular solution of $A\boldsymbol{x} = \boldsymbol{b}$ to get another solution).

11. If $\boldsymbol{b} = \boldsymbol{0}$, then there are infintely many solutions.

True. This is the $\boldsymbol{b} = \boldsymbol{0}$ case of question 10. There is at least one solution to $A\boldsymbol{x} = \boldsymbol{0}$, since $\boldsymbol{x} = \boldsymbol{0}$ is a solution, and then the set of solutions (i.e., the nullspace of A) is at least one-dimensional.

12. For all vectors $\boldsymbol{b}$ there is at least one solution.

False. Since the rank of A is less than n, and the set of vectors $\boldsymbol{b}$ for which there is a solution has dimension equal to the rank of A, there will be vectors in $\mathbb{R}^n$ for which the problem cannot be solved.