Math 240, Spring 2020 **Chenglong Yu**

Homework 2

DUE: THURSDAY, JANUARY 30

This week. Try to use Mathematica, Matlab or Sage if you haven't already. Read 2.1-2.2 and 2.6 in the book. Become comfortable parametrizing the solution space to a linear system once the reduced row echelon form is known.

Systems of Equations/Row Reduction

1. Which of the following matrices are in reduced row echelon form?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 219 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

2. Let the rows of A be $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$. Fnd a matrix B such that BA is the matrix

$$\begin{bmatrix} \mathbf{r}_1 + \mathbf{r}_2 \\ \mathbf{r}_2 - 2\mathbf{r}_3 \\ \mathbf{r}_1 + 3\mathbf{r}_3 - 5\mathbf{r}_2 \end{bmatrix}.$$

3. Compute (showing all your steps) the reduced row echelon form of the following matrix:

$$\begin{bmatrix} 2 & 4 & 1 & 11 & 14 \\ 1 & 2 & 1 & 8 & 10 \\ 4 & 8 & 2 & 22 & 28 \end{bmatrix}.$$

4. Compute the reduced row echelon form of the matrix A and find the general solution of the equation $A\mathbf{x} = 0$, where

$$A = \begin{bmatrix} 2020 & 2020 & 2020 & 2020 & 2020 \\ 2020 & 2020 & 2020 & 2020 & 2020 \\ 2020 & 2020 & 2020 & 2020 & 2020 \\ 2020 & 2020 & 2020 & 2020 & 2020 \\ 2020 & 2020 & 2020 & 2020 & 2020 \end{bmatrix}.$$

5. Solve the system of linear equations. Goode and Annin, Section 2.5, 7.

Matrix Inverses

6. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Check by direct computation that if $ac - bd \neq 0$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

7. Find the following matrix inverses using the formula for the inverse of a 2×2 matrix:

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

8. Compute via row-reduction the inverses of the following matrices, if they exist.

   a) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

   b) $\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$.

9. In your head, find the inverse of the matrix $\begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}$, where $a_i \neq 0$ for all i.

10. a) If A, D, and S are square matrices, S is invertible, and $A = SDS^{-1}$, show that $A^n = SD^nS^{-1}$.

    b) Compute $\begin{bmatrix} 17 & -6 \\ 35 & -12 \end{bmatrix}^5$ by hand using the equation

$$\begin{bmatrix} 17 & -6 \\ 35 & -12 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}.$$

11. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ 3 & 7 \end{pmatrix}$

    a) Show that A and B don't commute, i.e. that $AB \neq BA$

    b) Compute the matrix $(A + B)^2$

    c) Compute the matrix $A^2 + 2AB + B^2$

    d) Notice that your answers in part (b) and part (c) are different, i.e. apparently the familiar algebraic identity doesn't seem to hold for square matrices. Explain, in general, why this is the case, i.e. why in general $(A + B)^2 \neq A^2 + 2AB + B^2$. What conditions should the square matrices A and B satisfy for the identity to hold?

12. Determine whether the following statements are true or false. Justify your answer.

a) Let A be an $n \times n$ matrix such that $A^2 = 0$. Then the matrix $I_n - A$ is invertible with inverse $I_n + A$. (Here I_n denotes the $n \times n$ identity matrix.)

b) The only $n \times n$ matrices whose inverse is itself are I_n and $-I_n$

c) If a square matrix A is invertible, then its transpose A^T is also invertible.

d) The inverse A^{-1} of an invertible symmetric matrix A is also symmetric.

e) The product of two 2020×2020 matrices with rank 2020 is also of rank 2020.

f) The sum of two invertible matrices is also invertible.

g) If the $n \times n$ matrix A is not invertible, then the linear system $A\vec{x} = \vec{b}$ has infinitely many solutions for every $\vec{b} \in \mathbb{R}^n$

h) If the $n \times n$ matrix A in not invertible, then the homogenous linear system $A\vec{x} = 0$ has infinitely many solutions.