**Math 240, Spring 2020** **Chenglong Yu**

# Homework 3

Due: Thursday, February 6

**This week**. Read chapter 3, ignoring the material concerned with permutations, and chapter 4.1-4.3, especially page 253 definition 4.2.1 and page 259 theorem 4.2.7.

1. Let $A$ be the $n \times n$ matrix with $i, j$ entry $i + j$. What is the rank of $A$?

2. Compute the following determinants:

$$\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix}, \quad \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}, \quad \begin{vmatrix} 1 & \log_b a \\ \log_a b & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}.$$

3. Without doing any calculations, explain why $\begin{vmatrix} 1 & e & 1 & 1 \\ 2 & \pi & 2 & 1 \\ 3 & \sqrt{5} & 3 & 0 \\ 4 & 219 & 4 & 1 \end{vmatrix} = 0.$

4. a) By generalizing the method in class used to find the equation of a plane through three points, explain why the general equation for the equation for a circle through any three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ in the plane is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

   b) Use the method of the previous part to find the equation for the circle passing through $(1, 7)$, $(6, 2)$, and $(4, 6)$. Simplify your answer until it is in standard form.
   [Suggestion: compute the determinant by a cofactor expansion.]

5. True or False (Explain your answer briefly in either case)
   a) $\det(4A) = 4\det(A)$ for all $4 \times 4$ matrices $A$.
   b) $\det(A + B) = \det(A) + \det(B)$ for all $5 \times 5$ matrices $A$ and $B$.
   c) $\det(A^{10}) = (\det A)^{10}$ for all $10 \times 10$ matrices $A$.
   d) If all the entries of a $7 \times 7$ matrix $A$ are 7, then $\det A$ must be $7^7$.
   e) The equation $\det(-A) = \det A$ holds for all $6 \times 6$ matrices.
   f) The equation $\det(-A) = \det A$ holds for all $7 \times 7$ matrices.

g) If all the entries of a square matrix are 1 or 0, then $\det A$ must be $1, 0$ or $-1$.

6. Show that the matrix $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & -7 & -6 & 8 \\ -2 & 7 & 0 & -1 & -2 \\ -3 & 6 & 1 & 0 & -219 \\ -4 & -8 & 2 & 219 & 0 \end{bmatrix}$ is not invertible.
[HINT: The fast way to do this just uses that $A = -A^T$ and properties of determinants. ]

7. Compute $|A|^2$, where $A = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$.
[HINT: Last week you showed $A = A^{-1}$. Use properties of determinants and that $AA^{-1} = I$.]

8. Let $A$ be a $4 \times 4$ matrix with determinant 5. Give a proof or counterexample for each of the following.
   a) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.
   b) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
   c) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has no solution.
   d) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

9. Suppose that $A$ and $B$ are $4 \times 4$ invertible matrices. If $|A| = -2$ and $|B| = 3$, compute
$$\det(((A^{-1}B)^T(2B^{-1}))).$$

10. Draw pictures of subsets $A$ and $B$ of $\mathbb{R}^2$ which have the following properties.
   a) $A$ is closed under vector addition, but not under scalar multiplication.
   b) $B$ is closed under scalar multiplication, but not under vector addition.

11. Let $S = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = (r - 2s, 3r + s, s), r, s \in \mathbb{R}\}$.
   a) Show that $S$ is a subspace of $\mathbb{R}^3$.
   b) Show that the vectors in $S$ lie on the plane with equation $3x - y + 7z = 0$.

12. Which of the following sets are vector spaces? For the ones which are not, explain why.
   a) The set of solutions to the ODE $f'' + f' + 1 = 0$.
   b) The set of solutions to a matrix equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq 0$.

c) The set of solutions to the ODE $f'' + f' = 0$.

d) The set of solutions to a matrix equation $A\mathbf{x} = 0$.

e) The set of functions $u(x, y)$ defined on the unit square $[0, 1] \times [0, 1]$ satisfying $\Delta u =: \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$.

f) The set of all polynomials $p(x)$ satisfying $p(3) = p(4)$.

g) The set of polynomials of degree exactly 2020.