Homework 4

DUE: THURSDAY, FEBRUARY 13

This week. Read through 4.6, with an eye towards building intuition.

Differential Equations

1. The spring-mass system is subject to an additional force: a damping force proportional to the mass' velocity, that is $F_d = -cy'(t)$ for some constant $c > 0$. Newton's second law gives the following differential equation for the motion of the system:

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0. \tag{1}$$

   a) Show that the set of functions $V = \left\{f : \mathbb{R} \to \mathbb{R} : f'' + \frac{c}{m}f' + \frac{k}{m}f = 0\right\}$ is a vector space.
   b) In the case where $c/m = k/m = 2$, check that $f_1(t)$ and $f_2(t)$ solve (1), where

$$f_1(t) = e^{-t}\sin t \quad \text{and} \quad f_2(t) = e^{-t}\cos t.$$

   c) Using the preceding parts, find a solution $y(t)$ of (1) (when $c/m = k/m = 2$) with initial position and velocity given by $(y(0), y'(0)) = (2, 1)$.

2. Consider the equation

$$ru''(r) + u'(r) = 0. \tag{2}$$

   a) Show that the set of functions $V = \{f : \mathbb{R} \to \mathbb{R} : rf''(r) + f'(r) = 0\}$ is a vector space.
   b) Verify by direct substitution that $f_1(r) = 1$ and $f_2(r) = \ln r$ solve (2).
   c) Using the preceding parts, find a solution $u(r)$ of (2) satisfying $u(1) = 1$ and $u(2) = 2$.

   EDUCATIONAL REMARK: Equation (2) models the equilibrium temperature distribution in a circular pool (where r is the distance from the center) where the pool's outer edge is kept at a fixed temperature. You can learn more about this in Math 241.

Linear Independence and Bases

3. Show that $\{e^{-t}\sin t, e^{-t}\cos t\}$ is linearly independent in the vector space V from problem 1.

4. Is the set $\{\sin 2x, \cos 2x, \cos x \sin x\}$ linearly independent in the space of all functions?

5. Find $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$, where $\mathbf{v}_1 = (1, 2, 3, 4)^T$ and $\mathbf{v}_2 = (5, 6, 7, 8)^T$. Your answer should consist of a set of equations, which determines whether a vectors (x, y, z, t) is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$.

6. Find *all* solutions of the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$

where $\mathbf{v}_1 = (1,1,0)^T$, $\mathbf{v}_2 = (0,1,1)^T$, and $\mathbf{v}_3 = (1,0,1)^T$. What conclusion can you make about linear independence (dependence) of the system of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

7. Determine a basis for the nullspace (kernel) of A, where $A = \begin{bmatrix} 2 & 4 & 1 & 11 & 14 \\ 1 & 2 & 1 & 8 & 10 \\ 4 & 8 & 2 & 22 & 28 \end{bmatrix}$.

8. Write down a basis for the vector space of 4×4 symmetric matrices. You do not need to prove your set is a basis.

9. Determine whether the set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

10. Determine whether the given set of vectors is linearly independent in $M_{2\times 2}(\mathbb{R})$:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 3 & 6 \\ 0 & 1 \end{bmatrix}.$$

11. Spring 2016 Final, Problem 8 (a). See:
https://www.math.upenn.edu/ugrad/calc/m240/exams/240spring16final.pdf

12. Spring 2016 Final, Problem 3. See:
https://www.math.upenn.edu/ugrad/calc/m240/exams/240spring16final.pdf

13. Goode and Annin: 4.4.10

14. Goode and Annin: 4.6.8