Math 240, Spring 2020 **Chenglong Yu**

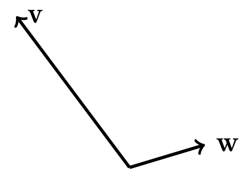
Homework 5

DUE: TUESDAY, FEBRUARY 25

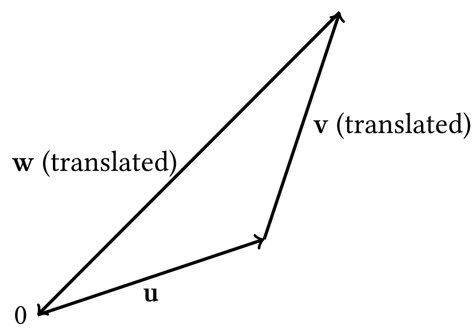
This week. Read through 4.9. Intuitively understand the difference between a vector and its coordinate vector with respect to some basis.

Bases and Coordinate Vectors

1. In the accompanying figure, sketch the vector $\mathbf{u}$ with $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, where $\mathcal{B}$ is the basis for $\mathbb{R}^2$ consisting of the vectors $\mathbf{v}, \mathbf{w}$.



2. Consider the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ sketched in the accompanying figure. Find the coordinate vector of $\mathbf{w}$ with respect to the basis $\mathbf{u}, \mathbf{v}$.



3. Let $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ be the standard ordered basis for $\mathbb{R}^4$ and let $C = \{e_3, e_1, e_2, e_4\}$. Compute the change of bases matrices $P_{C\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow C}$.

4. Consider the set $V = \left\{f : \mathbb{R} \to \mathbb{R} : f(x) = \frac{c_1 x + c_2}{(x-1)(x-2)} \text{ for some } c_1, c_2 \in \mathbb{R}\right\}$.
   a) Show that V is a subspace of the space of all functions.
   b) One basis for V is $\mathcal{B} = \left\{\frac{1}{(x-1)(x-2)}, \frac{x}{(x-1)(x-2)}\right\}$. Another is $C = \left\{\frac{1}{x-1}, \frac{1}{x-2}\right\}$. Compute the change of basis matrix $P_{C\leftarrow\mathcal{B}}$.
   c) Using part b), the fact that
   $$\int \frac{A}{x-1} + \frac{B}{x-2}\, dx = A \log|x-1| + B\log|x-2| + C,$$
   and the equation $[\mathbf{v}]_C = P_{C\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$, effortlessly compute the following integrals:
   $$\int \frac{2x+1}{(x-1)(x-2)} dx, \qquad \int \frac{3x-8}{(x-1)(x-2)} dx.$$

5. For the problems below, determine the component vector of the given vector in the vector space V relative to the given ordered basis $\mathcal{B}$.

a) $V = \mathbb{R}^3$; $\mathcal{B} = \{(1,0,1), (1,1,-1), (2,0,1)\}$; $\mathbf{v} = (-9, 1, -8)$.

b) $V = P_2(\mathbb{R})$; $\mathcal{B} = \{5 - 3x, 1, 1 + 2x^2\}$; $p(x) = 15 - 18x - 30x^2$.

c) $V = M_{2\times 2}(\mathbb{R})$; $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$; $A = \begin{bmatrix} -3 & -2 \\ -1 & 2 \end{bmatrix}$.

The Rank Nullity Theorem

6. Use the reduced row echelon form computed in HW2 to find a basis for colspace(A), where

$$A = \begin{bmatrix} 2 & 4 & 1 & 11 & 14 \\ 1 & 2 & 1 & 8 & 10 \\ 4 & 8 & 2 & 22 & 28 \end{bmatrix}.$$

7. For the problems below, determine the nullity of A "by inspection" by appealing to the Rank-Nullity Theorem. Avoid computations.

a) $A = \begin{bmatrix} 2 & -3 \\ 0 & 0 \\ -4 & 6 \\ 22 & -33 \end{bmatrix}$.

b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

c) $A = \begin{bmatrix} 0 & 0 & 0 & -2 \end{bmatrix}$.

Differential Equations

8. Consider the set $V = \{f : \mathbb{R} \to \mathbb{R} : f'' - f = 0\}$ of solutions to the ODE $f'' = f$.

a) Show that e^x, e^{-x}, $\sinh x$, and $\cosh x$ are all elements of V.

b) Show that the lists of vectors $\{e^x, e^{-x}\}$ and $\{\sinh x, \cosh x\}$ are each linearly independent.

c) It can be shown that V is a two-dimensional vector space (accept this for now), so that $\mathcal{B} = \{\cosh x, \sinh x\}$ and $\mathcal{C} = \{e^x, e^{-x}\}$ are bases for V.

d) Let $u(x)$ be the solution to the initial value problem

$$u''(x) - u(x) = 0, \qquad u(0) = 5, \quad u'(0) = -2.$$

Pick one of the bases above and write u as a linear combination of the basis elements.

Comment: The calculations in d) are noticeably simpler in one of the bases—that's the point!

9. Recall the equation (HW4) for the position $y(t)$ of a spring-mass system with a damping force:

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0, \tag{1}$$

where $c > 0$ is the damping constant and $k > 0$ is the spring constant. Let $E(t) = \frac{1}{2}my'^2 + \frac{1}{2}ky^2$ be the total (potential plus kinetic) energy.

a) Without solving the differential equation, show that $E'(t) \leq 0$.

b) Use this to show that if $y(0) = 0$ and $y'(0) = 0$, then $y(t) = 0$ for all $t \geq 0$.

c) Suppose that $u(t)$ and $v(t)$ both satisfy equation (1), and also $u(0) = v(0)$ and $u'(0) = v'(0)$. Show that $u(t) = v(t)$ for all $t \geq 0$.

d) Show that the set of functions $V = \left\{f : \mathbb{R} \to \mathbb{R} : f'' + \frac{c}{m}f' + \frac{k}{m}f = 0\right\}$ is a vector space.