Math 240, Spring 2020 **Chenglong Yu**

Homework 6

DUE: THURSDAY, MARCH 5

This week. Read Chapter 6, with an eye towards answering the following question: Why are we reserving special attention to Linear Transformations, instead of studying more general functions as you did in 114?

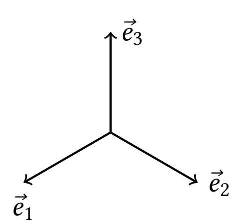
Linear Transformations

1. For the problems below, verify directly from the definition that the giving function is a linear transformation.
   a) $T : C^2([a,b]) \to C^0([a,b])$ defined by $T(f) = f'' - 16f$. (Here $C^2([a,b])$ is the vector space of functions defined on $[a,b]$ with continuous second order derivatives, $C^0([a,b])$ is the vector space of continuous functions.
   b) $T : M_{n\times n} \to M_{n\times n}$ defined by $T(A) = AB - BA$, where B is a fixed $n \times n$ matrix.
   c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)$, where $\theta \in \mathbb{R}$ is fixed.

2. For the problems below, determine the matrix of the given transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.
   a) $T(x_1, x_2, x_3) = (x_3 - x_1, -x_1, 3x_1 + 2x_3, 0)$.
   b) $T(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)$, where $\theta \in \mathbb{R}$ is fixed.

3. In this problem, let V be the vector space of 2×2 matrices with real entries, and let $T : V \to \mathbb{R}$ be defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$, in other words TM is the trace of M.
   a) Show that T is a linear transformation.
   b) Compute the matrix $[T]_{\mathcal{B}}^{C}$ associated to the linear transformation T, where
   $$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \quad C = \{1\}.$$
   c) Compute the rank and nullity of the matrix $[T]_{\mathcal{B}}^{C}$.

4. Find the matrix with respect to the standard basis for the (right hand oriented) rotation by 120 degrees in $\mathbb{R}^3$ about the vector $(1,1,1)$. The accompanying picture may be helpful. Note: the answer is simple—involved computations are unnecessary.



5. Find a 2×2 matrix A (other than $A = I_2$) such that $A^7 = I_2$.

6. a) Find a 2×2 matrix that rotates the plane by $+45$ degrees.
   b) Find a 2×2 matrix that reflects across the horizontal axis.
   c) Find a 2×2 matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.
   d) Find a 2×2 matrix that reflects across the horizontal axis followed by a rotation in the plane by $+45$ degrees.
   e) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.
   f) Find the inverse of each of these maps.

7. Find 3×3 matrices A and B which act on $\mathbb{R}^3$ as follows.
   a) A keeps the x_1 axis fixed but rotates the x_2x_3 plane by 60 degrees.
   b) B rotates the x_1x_3 plane by 60 degrees and leaves the x_2 axis fixed.

8. Consider the linear transformation T from $U^{2\times 2}$, the space of upper triangular 2×2 matrices, to $U^{2\times 2}$, given by $T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^{-1} M \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Let A be the $\mathcal{C}$-matrix of T, B be the $\mathcal{B}$-matrix of T, and $P = P_{\mathcal{C}\leftarrow\mathcal{B}}$ the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$, where $\mathcal{C}$ and $\mathcal{B}$ are the bases of $U^{2\times 2}$ given by

$$\mathcal{C} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \quad \text{and} \quad \mathcal{B} := \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

   a) Find A and B.
   b) State a known relation between A, B, and P.
   c) Find S and verify that the relation you stated in b) above holds.

9. True or false: If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space V and $T : V \to W$ is a linear transformation. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\{T\mathbf{u}, T\mathbf{v}, T\mathbf{w}\}$ is linearly independent.

10. For the transformation of $T : \mathbb{R}^2 \to \mathbb{R}^2$ with the given matrix, sketch the image of the square with vertices $(1, 1), (2, 1), (2, 2)$, and $(1, 2)$.
    a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
    b) $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

d) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

11. Which of the following maps F are linear?
   a) $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(x, y, z) = (x, z)$.
   b) $F : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $F(\mathbf{x}) = -\mathbf{x}$.
   c) $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(\mathbf{x}) = \mathbf{x} + (0, -1, 0)$.
   d) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (3x + y, y)$.
   e) $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) = xy$.

Applied Problems

12. [Application to genetics.] In autosomal inheritance, each individual inherits one gene from each of its parents' pairs of genes to form its own particular pair. It is believed that each of a parent's two genes are equally likely to be passed on to the offspring. The following table lists the probabilities of the possible genotypes of the offspring for all the possible combinations of the parents' genotypes.

		Parent's Genotype					
		AA-AA	AA-Aa	AA-aa	Aa-Aa	Aa-aa	aa-aa
Offspring Genotype	AA	1	1/2	0	1/4	0	0
	Aa	0	1/2	1	1/2	1/2	0
	aa	0	0	0	1/4	1/2	1

A breeder has a large population of dogs consisting of some distribution of all three genotypes *AA*, *Aa*, and *aa*. By design, each dog in the population mates with a dog of the same genotype. Your goal is to determine the distribution of each genotype in any given generation. For $n = 0, 1, 2, \ldots$, let

$$\begin{aligned} a_n &= \text{fraction of dogs of genotype AA in the } n\text{-th generation,} \\ b_n &= \text{fraction of dogs of genotype Aa in the } n\text{-th generation,} \\ c_n &= \text{fraction of dogs of genotype aa in the } n\text{-th generation.} \end{aligned}$$

Define the vector $\mathbf{x}^{(n)} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$.

   a) Find a matrix M such that $\mathbf{x}^{(n)} = M\mathbf{x}^{(n-1)}$ (for $n = 1, 2, \ldots$).

b) Since $\mathbf{x}^{(n)} = M\mathbf{x}^{(n-1)} = M^2\mathbf{x}^{(n-2)} = \cdots = M^n\mathbf{x}^{(0)}$, it is desirable to compute large powers of M. Using Mathematica, compute M^3, M^6, and M^9. Does it appear that M^n approaches a limit? If so, what do you conjecture $\lim_{n\to\infty} \mathbf{x}^{(n)}$ to be?