Homework 7

Due: Thursday, March 19

This week. Read chapter 7.

Rank-Nullity theorem and ODE

1. Let $V = \left\{ f : \mathbb{R} \to \mathbb{R} : f'' + \frac{c}{m} f' + \frac{k}{m} f = 0 \right\}$ be the vector space of functions satisfying the differential equation for the damped spring-mass system (cf. HW5).
   a) Show that the function $T : V \to \mathbb{R}^2$ by $T(f) = \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix}$ is a linear transformation.
   b) Show that $\ker T = \{\mathbf{v} \in V : T\mathbf{v} = 0\}$ consists only of the zero function (you will need to use the result of part c) from the problem in HW5).
   c) Show that V is *at most* two dimensional by Rank-Nullity theorem.

   Comment: c) means the set of solutions is not too big. In particular, once two linearly independent solutions are known, by taking linear combinations, we have found *all* of the solutions!

Eigenvalues and Eigenvectors

2. True or False (Explain your answer briefly in either case)
   a) If two matrices A and B have the same characteristic polynomial, then A and B have exactly the same set of eigenvalues.
   b) If two matrices A and B have the same characteristic polynomial, then A and B have exactly the same set of eigenvectors.
   c) Each eigenspace of an $n \times n$ matrix is a subspace of $\mathbb{R}^n$.
   d) It is possible for a square matrix A to have infinitely many different eigenvectors.
   e) A linear combination of a set of eigenvectors of a matrix A is again an eigenvector of A.
   f) If a matrix A has a repeated eigenvalue, then it is defective.
   g) An $n \times n$ matrix A is nondefective if it has n different eigenvectors.
   h) If $\mathbf{v}$ is an eigenvector of A, then $\mathbf{v}$ is also an eigenvector of A^5.
   i) If A^2 has eigenvalue 9, must 3 or -3 be an eigenvalue of A?
   j) The eigenvalues of A and A^T are the same.
   k) 0 can be an eigenvalue of a matrix.

3. Find all eigenvalues as well as the dimension of each eigenspace for the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

4. Write down a simple 3×3 matrix with eigenvalues $3, 4$, and 7.

5. The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflects each vector across the line $y = x$. By arguing geometrically, determine all eigenvalues and eigenvectors of A.

6. Determine all eigenvalues and corresonding eigenvectors of the given matrix.

   a) $\begin{bmatrix} 1 & 6 \\ 2 & -3 \end{bmatrix}$.

   b) $\begin{bmatrix} 7 & -8 & 6 \\ 8 & -9 & 6 \\ 0 & 0 & -1 \end{bmatrix}$

   c) $\begin{bmatrix} 1 & 1/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/4 & 1 \end{bmatrix}$.

Diagonalization

7. Determine whether the given matrix A is diagonalizable. Where possible, find a matrix S such that $S^{-1}AS$ is diagonal, with the eigenvalues along the diagonal.

   a) $\begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix}$.

   b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 7 \\ 1 & 1 & -3 \end{bmatrix}$.

8. Is the matrix $A = \begin{bmatrix} 2020 & 2020 & 2020 & 2020 & 2020 \\ 2020 & 2020 & 2020 & 2020 & 2020 \\ 2020 & 2020 & 2020 & 2020 & 2020 \\ 2020 & 2020 & 2020 & 2020 & 2020 \\ 2020 & 2020 & 2020 & 2020 & 2020 \end{bmatrix}$ diagonalizable? If so, exhibit matrices S and D, where D is diagonal, such that $A = SDS^{-1}$. Otherwise, explain why A is defective.

Applied Problems

9. This is a continuation of the genetics problem from HW6. Let M be the matrix given there.
   a) Diagonalize M.
   b) For any $n \in \mathbb{N}$, compute all the entries in M^n.
   c) Evaluate $\lim_{n\to\infty} M^n$ and comment on the limiting distribution $\lim_{n\to\infty} \mathbf{x}^{(n)}$.