HOMEWORK 1,2 SOLUTION

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1 Problem 1

Only the second matrix and the fifth matrix are in reduce row echelon form.

2 Problem 2

Let

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & -5 & 3 \end{pmatrix}$$

Then writing $r_i = (r_{i1}\ r_{i2}\ \cdots)$, we get

$$BA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & -5 & 3 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & \cdots \\ r_{21} & r_{22} & \cdots \\ r_{31} & r_{32} & \cdots \end{pmatrix} = \begin{pmatrix} r_{11} + r_{21} & r_{12} + r_{22} & \cdots \\ r_{21} - 2r_{31} & r_{22} - 2r_{32} & \cdots \\ r_{11} - 5r_{21} + 3r_{31} & r_{12} - 5r_{22} + 3r_{32} & \cdots \end{pmatrix} = \begin{pmatrix} r_1 + r_2 \\ r_2 - 2r_3 \\ r_1 - 5r_2 + 3r_3 \end{pmatrix}$$

This is the most explicit way to do this problem.

3 Problem 3

Row reduce as usual

$$\begin{pmatrix} 2 & 4 & 1 & 11 & 14 \\ 1 & 2 & 1 & 8 & 10 \\ 4 & 8 & 2 & 22 & 28 \end{pmatrix} \xrightarrow{r_3 = r_3 - 2r_1} \begin{pmatrix} 2 & 4 & 1 & 11 & 14 \\ 1 & 2 & 1 & 8 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 = 2r_2} \begin{pmatrix} 2 & 4 & 1 & 11 & 14 \\ 2 & 4 & 2 & 16 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_2 = r_2 - r_1} \begin{pmatrix} 2 & 4 & 1 & 11 & 14 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_1 = r_1 - r_2} \begin{pmatrix} 2 & 4 & 0 & 6 & 8 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_1 = r_1/2} \begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4 Problem 4

If you row reduce A, you get a matrix whose first row is $(1, 1, 1, 1, 1)$ and all other rows zero vectors. So only the first row gives us a useful relation, which is:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0 \iff x_1 = -x_2 - x_3 - x_4 - x_5$$

showing that the general solution is of the form

$$\begin{pmatrix} -x_2 - x_3 - x_4 - x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

where x_2, x_3, x_4, x_5 are free variables that could take any value in $\mathbb{R}$.

5 Problem 5

6 Problem 6

We check that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{pmatrix} = \begin{pmatrix} ad/(ad-bc) - bc/(ad-bc) & -ab/(ad-bc) + ab/(ad-bc) \\ cd/(ad-bc) - cd/(ad-bc) & -bc/(ad-bc) + ad/(ad-bc) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly for the other way

$$\begin{pmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad/(ad-bc) - bc/(ad-bc) & db/(ad-bc) - bd/(ad-bc) \\ -ca/(ad-bc) + ac/(ad-bc) & -bc/(ad-bc) + ad/(ad-bc) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

7 Problem 7

Here are the inverse matrices in order:

$$-1 \cdot \begin{pmatrix} 7 & -3 \\ -5 & 2 \end{pmatrix}, \quad \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad -\frac{1}{2} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

8 Problem 8

a)

$$\begin{pmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 5/3 & * & * & * \\ 0 & 1 & -1/3 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{pmatrix}$$

so the left submatrix cannot be the identity matrix implying that the original matrix is not invertible.

b) We do the same thing

$$\begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & -1/2 & -1/2 & 0 & 1 & 0 & 0 \\ 1/2 & -1/2 & 1/2 & -1/2 & 0 & 0 & 1 & 0 \\ 1/2 & -1/2 & -1/2 & 1/2 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1 & 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

so we see that the original matrix is in fact its own inverse. We'll maybe see that this is because columns are orthonormal and the matrix is symmetric.

9 Problem 9

The answer is

$$\begin{pmatrix} a_1^{-1} & 0 & \cdots & 0 \\ 0 & a_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^{-1} \end{pmatrix}$$

10 Problem 10

a) Clearly true for $n = 1$. Suppose true for n. Then we have

$$A^{n+1} = A^n \cdot A = SD^nS^{-1}SDS = SD^{n+1}S^{-1}$$

by induction.

b) Using part a), we get, since two matrices on the side are inverse of each other

$$\begin{pmatrix} 17 & -6 \\ 35 & -12 \end{pmatrix}^5 = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^5 \begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 32 & 0 \\ 0 & 243 \end{pmatrix} \begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 3197 & -1266 \\ 7385 & -2922 \end{pmatrix}$$

11 Problem 11

a) Check that

$$AB = \begin{pmatrix} 10 & 17 \\ 15 & 16 \end{pmatrix}, \quad BA = \begin{pmatrix} 13 & 11 \\ 24 & 13 \end{pmatrix}$$

b) Compute

$$(A+B)^2 = \begin{pmatrix} 25 & 25 \\ 36 & 64 \end{pmatrix}$$

c) Compute

$$A^2 + 2AB + B^2 = \begin{pmatrix} 45 & 48 \\ 63 & 78 \end{pmatrix}$$

d) In general they are different because we have $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$. So we need $AB = BA$ for the identity to hold

12 Problem 12

a) **TRUE**. Check $(I_n - A)(I_n + A) = I_n - A + A + A^2 = I_n$ and the other way too. So A is invertible.
b) **False**. The following matrix has itself as the inverse:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

c) **True**. If you know some properties of the determinant, this is clear. Applying row operations to A is the same as applying column operations to A^T, so A^T has the same determinant as A. So A is invertible if and only if A^T is.
d) **True** We have $A^T = A$. So the question is whether $(A^{-1})^T = A^{-1}$. We check whether $(A^{-1})^T$ is the inverse to A, from which we conclude that $A^{-1} = (A^{-1})^T$, since the inverse is unique.

$$A \cdot (A^{-1})^T = A^T \cdot (A^{-1})^T = (A^{-1}A)^T = I^T = I$$

e) **TRue** Such a matrix has full rank, thus has nonzero determinant. Since determinant is multiplicative, that is, $\det(AB) = \det(A)\det(B)$ whenever defined, we conclude that if both A and B have full rank, nonzero determinant, then so does their product AB.
f) **False** $I_n + (-I_n) = 0$ is clearly not invertible.
g) **False** If you think about the reduced row echelon form, such a systm has either infinitely many solutions or no solution by making a false relation such as $(0,0) \cdot (x,y) = 1$.
h) **True** Since the column vector b is the zero vector, we never run into a false relation, so such system always has infinitely many solutions.