Homework 3 Solutions

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1. Matrix A has the form:

$$\begin{bmatrix} 2 & 3 & 4 & \dots & n+1 \\ 3 & 4 & 5 & \dots & n+2 \\ 4 & 5 & 6 & \dots & n+3 \\ \vdots & \vdots & \vdots & \ddots & \\ n+1 & n+2 & n+3 & & n+n \end{bmatrix}$$

   which can be row reduced to form $n-2$ rows of zeros. Then $\text{rank}(A) = 2$.

2. (a) $\det(A) =$ 15-14=1
   (b) $\det(A) = \cos^2(\alpha) + \sin^2(\alpha) = 1$
   (c) $\det(A) = 1 - \log_b(a) \cdot \log_a(b) = 0$
   (d) By subtracting row 1 from each of the three rows below, A becomes upper triangular with diagonal entries 1, -2, -2, and -2. Remember that adding rows to each other does not effect the value of the determinant. Then $\det(A) = -8$.
   (e) By using cofactor expansion along the bottom row, $\det(A) = -abcd$.

3. Since $\det(A^T) = \det(A)$ and A^T has two equivalent rows, $\det(A)$=0.

4. (a) The general equation of a circle is $(x-h)^2 + (y-k)^2 = r^2$, which reads $(x^2+y^2) - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$ when fully expanded. Since h, k, r are the geometric properties of the circle, it is more convenient to express the equation in terms of constants c: $c_1(x^2+y^2) + c_2x + c_3y + c_4 = 0$. Since the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ must satisfy the equation of the circle they lie on, we can form the following matrix equation to find constants c:

$$\begin{bmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

   We know that $c_1 = c_2 = c_3 = c_4 = 0$ is a solution. However, we know at least one circle exists, so we know that there is at least one nonzero solution to the above equation. This makes at least two solutions and thus (since a system either has zero, one, or infinitely many solutions) infinitely many solutions. We know that if a system has infinitely many solutions, its determinant will be zero. Solving the determinant shown in the problem gives you the coefficients c_1, c_2, c_3, c_4 that define the circle passing through those three points.
   (b) By computing the determinant using cofactor method, the equation is:
   $(x^2+y^2) - 2x - 4x - 20 = 0$, which can be simplified by completing the square into:
   $(x-1)^2 + (y-2)^2 = 25$.

5. (a) False, $\det(4A) = 4^n \det A = 256 \cdot \det(A)$
   (b) False, in general, $\det(A+B) \neq \det(A) + \det(B)$.
   (c) True, $\det(A^n) = (\det(A))^n$.
   (d) False, $\det(A)$=0 since there are at least two equivalent rows.
   (e) True, $\det(-A) = (-1)^n \det(A)$, so for n even, det(-$A$) = $\det(A)$

(f) False, $\det(-A) = (-1)^n \det(A)$, so for n odd, det(-$A$) $= -\det(A)$

(g) False, consider $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$ as a counterexample.

6. Since $\det(A)$=0, A is not invertible:

$$\begin{aligned} \det(A) &= \det(A^T) \\ &= \det(-A) \\ &= (-1)^5 \det(A) \\ \det(A) &= -\det(A) \\ 2\det(A) &= 0 \\ \det(A) &= 0 \end{aligned}$$

7. $(\det(A))^2 = \det(AA) = \det(AA^{-1}) = \det(I) = 1$

8. (a) Since $\det(A)$ is nonzero, A has one unique solution.

   (b) For A to have infinitely many solutions, $\text{rank}(A) < n$. Since $\det(A) \neq 0$, $\text{rank}(A) =$ n.

   (c) For A to have infinitely many solutions, $\text{rank}(A) < n$ and $\text{rank}(A) < \text{rank}(b)$. Again, since $\det(A) \neq 0$, $\text{rank}(A) =$ n.

   (d) From (a), since $Ax = b$ has one unique solution, $Ax = b$ has at least one solution. If $b = 0$, this unique solution is the trivial solution $x = 0$.

9.

$$\begin{aligned} \det\Big(\big((A^{-1}B)^T(2B^{-1})\big)\Big) &= \det((A^{-1}B)^T) \cdot \det(2B^{-1}) \\ &= \det(A^{-1}B) \cdot 2^4 \det(B^{-1}) \\ &= 16\det(A^{-1})\det(B)\det(B^{-1}) \\ &= 16\det(A^{-1}) \\ &= -16(\frac{1}{2}) = -8 \end{aligned}$$

10. (a) One example of A is the positive real numbers $[0, \infty)$.

    (b) One example of B is the x-axis union the y-axis.

11. (a) We will show that S is closed under addition, closed under scalar multiplication, and contains the zero vector.

    i. Consider $v_1 \in S$ and $v_2 \in S$. We want to show that $v_1 + v_2 \in S$.
    $v_1 = (r_1 - 2s_1, 3r_1 + s_1, s_1)$ and $v_2 = (r_2 - 2s_2, 3r_2 + s_2, s_2)$
    $v_1 + v_2 = (r_1 - 2s_1 + r_2 - 2s_2, 3r_1 + s_1 + 3r_2 + s_2, s_1 + s_2)$
    $v_1 + v_2 = ((r_1 + r_2) - 2(s_1 + s_2), 3(r_1 + r_2) + (s_1 + s_2), (s_1 + s_2))$, which is in the form of a vector in S. Then S is closed under addition.

    ii. Consider $k \in \mathbb{R}$ and $v \in S$. We want to show that $kv \in S$.
    $v = (r - 2s, 3r + s, s)$
    $kv = k(r - 2s, 3r + s, s) = (kr - 2ks, 3kr + ks, ks)$. Since $r, s \in \mathbb{R}$, then also $kr, ks \in \mathbb{R}$. Thus we may scale our values of r, s and remain in S. S is closed under scalar multiplication.

    iii. 0 is an element; choose $r = 0, s = 0$.

    (b) It can be seen that $3(r - 2s) - (3r + s) + 7(s) = 0$.

12. It is important to check each of the ten properties of vector spaces when determining if a set is a vector space or not. In these solutions, for the sake of me not typing in LaTeX all night, only the critical properties are examined.

(a) Not a vector space because it is not closed under addition.
Given $f_1, f_2 \in S$, then $f_1'' + f_1' + 1 = 0$ and $f_2'' + f_2' + 1 = 0$.
If S is closed under addition, then $f_1 + f_2 \in S$.
$(f_1, f_2)'' + (f_1, f_2)' + 1 = 0$
$f_1'' + f_1' + f_2'' + f_2' + 1 = 0$. Since $f'' + f' = -1$,
$-1 + -1 + 1 \neq 0$. Then S is not closed under addition.

(b) Not a vector space because it is not closed under addition.
Given x_1, x_2 as solutions to $Ax = b$, then $Ax_1 = b$ and $Ax_2 = b$.
If S is closed under addition, then $A(x_1 + x_2) = b$.
$A(x_1 + x_2) = Ax_1 + Ax_2 = b + b = 2b \neq b$ unless $b = 0$. Since $b \neq 0$, this set is not closed under addition.

(c) A vector space. See (a). If $f'' + f' = 0$, S would be closed under addition, which was the property restricting S from being a vector space.

(d) A vector space. See (b). If b is allowed to be zero, then the set is closed under addition, which was the property restricting S from being a vector space.

(e) A vector space. Since taking derivatives is a linear operator, it is easy to check that S is closed under addition and scalar multiplication. Choosing $u(x, y) = 0$ satisfies the zero vector requirement.

(f) A vector space. Since $(p_1 + p_2)(3) = p_1(3) + p_2(3)$, it is easy to check that S is closed under addition and scalar multiplication. Choosing $p(x) = 0$ satisfies the zero vector requirement. Note that the zero function evaluated at any point is zero, namely $0(x) = 0$ for any x.

(g) Not a vector space. 0 is not a polynomial of degree 2020.