HOMEWORK 4 SOLUTIONS

JONGWON KIM

1 Problem 1

a) Taking $f(x) = 0$, the set V is clearly none empty. Let $f, g \in V$ and $r \in \mathbb{R}$. Then

$$(f+g)'' + \frac{c}{m}(f+g)' + \frac{k}{m}(f+g) = f'' + \frac{c}{m}f' + \frac{k}{m}f + g'' + \frac{c}{m}g' + \frac{k}{m}g = 0$$
$$(rf)'' + \frac{c}{m}(rf)' + \frac{k}{m}(rf) = r(f'' + \frac{c}{m}f' + \frac{k}{m}f) = 0$$

so V is closed under addition and scalar multiplication, thus a vector space.

b) In this case the system becomes

$$f'' + 2f' + 2f = 0$$

We easily check for $f_1 = e^{-t}\sin(t)$,

$f_1'' + 2f_1' + 2f_1 = (-e^{-t}\cos(t) - e^{-t}\sin(t) + e^{-t}\sin(t) - e^{-t}\cos(t)) + 2(e^{-t}\cos(t) - e^{-t}\sin(t)) + 2e^{-t}\sin(t) = 0$

The case where $f_2 = e^{-t}\cos(t)$ is done the same way.

c) You wil learn soon, but a general solution looks like $af_1 + bf_2$. So the initial condition $(f(0), f'(0)) = (2, 1)$ means

$$af_1(0) + bf_2(0) = 2 \iff b = 2$$
$$af_1'(0) + bf_2'(0) = 1 \iff a - b = 1$$

forcing that $f(t) = 3f_1 + 2f_2$

2 Problem 2

This problem is done the same way as problem 1

3 Problem 3

Suppose for some $a, b \in \mathbb{R}$ we have $ae^{-t}\sin(t) + be^{-t}\cos(t) = 0$. Then letting $t = 0$ and $t = \pi/2$ we get

$$b = 0$$
$$ae^{-\pi/2} = 0$$

but $e^{-\pi/2} \neq 0$, so $a = 0$ as well. Hence $\{e^{-t}\cos(t), e^{-t}\sin(t)\}$ is linearly independent over $\mathbb{R}$.

4 Problem 4

Recall the double angle formula

$$\sin(2x) = 2\sin(x)\cos(x)$$

So we see that we have a nontrivial linear combination equaling zero:

$$\sin(2x) + 0 \cdot \cos(2x) - 2\sin(x)\cos(x) = 0$$

showing that the given set is not linearly independent.

5 Problem 4

For $w = (a, b, c, d)^T$ to be in the span of v_1 and v_2, it means that there exists $x, y \in \mathbb{R}$ such that

$$xv_1 + yv_2 = w$$

in terms of a matrix equation,

$$\begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

has a solution. So we solve the above system of equation,

$$\begin{pmatrix} 1 & 5 & a \\ 2 & 6 & b \\ 3 & 7 & c \\ 4 & 8 & d \end{pmatrix} \xrightarrow{r_i = r_i - ir_1} \begin{pmatrix} 1 & 5 & a \\ 0 & -4 & b-2a \\ 0 & -8 & c-3a \\ 0 & -12 & d-4a \end{pmatrix} \xrightarrow{r_3 = r_3 - 2r_2, r_4 = r_4 - 3r_2} \begin{pmatrix} 1 & 5 & a \\ 0 & -4 & b-2a \\ 0 & 0 & a-2b+c \\ 0 & 0 & 2a-3b+d \end{pmatrix}$$

So we get $a - 2b + c = 0$ and $2a - 3b + d = 0$, two independent relations that define the span of v_1 and v_2. So we have

$$\begin{aligned} \text{span}\{v_1, v_2\} &= \{(a, b, c, d)^T \mid a - 2b + c = 0, 2a - 3b + d = 0\} \\ &= \{(a, b, -a + 2b, -2a + 3b) \mid a, b \in \mathbb{R}\} \end{aligned}$$

6 Problem 6

As in problem 5, we reduce the matrix whose columns are v_i's and the last column the zero vector

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So there is only the unique solution $(c_1, c_2, c_3) = (0, 0, 0)$, showing that $\{v_1, v_2, v_3\}$ is linearly independent.

7 Problem 7

As above, row reduce (looks familiar?)

$$A \to \begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Two nonzero rows give us relations

$$\begin{aligned} x_1 + 2x_2 + 3x_4 + 4x_5 &= 0 \\ x_3 + 5x_4 + 6x_5 &= 0 \end{aligned}$$

so the general solution is of the form

$$\begin{pmatrix} -2x_2 - 3x_4 - 4x_5 \\ x_2 \\ -5x_4 - 6x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 4 \\ 0 \\ -6 \\ 0 \\ 1 \end{pmatrix}$$

getting the following basis for the nullspace of A

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -6 \\ 0 \\ 1 \end{pmatrix} \right\}$$

8 Problem 8

Let E_{ij} denote the matrix whose entries are all 0 except that the (i, j)-th entry is 1. Here is one example of a basis

$$\{E_{11}, E_{22}, E_{33}, E_{44}, E_{12} + E_{21}, E_{13} + E_{31} E_{23} + E_{32}\}$$

9 Problem 9

Since there are three vectors, this set is a basis if and only if it is linearly independent. So check that

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \to I_3$$

showing that it is linearly independent and thus a basis.

10 Problem 10

check whether there is a nontrivial solution to $aA_1 + bA_2 + cA_3 = 0$, which rewrites

$$\begin{aligned} a + 2b + 3c &= 0 \\ a - b + 6c &= 0 \\ a + b + c &= 0 \end{aligned}$$

equivalent to solving

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 6 \\ 1 & 1 & -1 \end{pmatrix} \to I_3$$

so they are linearly independent.

11 Problem 11

Denote the given matrices by A, B, C. Check that $C - (A - B)$ is $3E_{22}$. Also, $A - B$ is $E_{12} + 2E_{21}$. Their span has elements $E_{22}, E_{12} + 2E_{21}, E_[11 + E_{22}$ which are clearly linearly independent. Dimension of W is at most 4 since it's a subspace of $M_{2\times 2}(\mathbb{R})$, but we now that it cannot be 4 since it is a proper subspace, forcing its dimension to be 3 as $\text{span}(C) \leq W$ has dimension 3. So W actually equals the span of C and so C is a basis.

12 Problem 12

We solve

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

showing that $s_1 = -1$, $s_2 = 2$, $s_3 = 1$ and so their sum is 2.