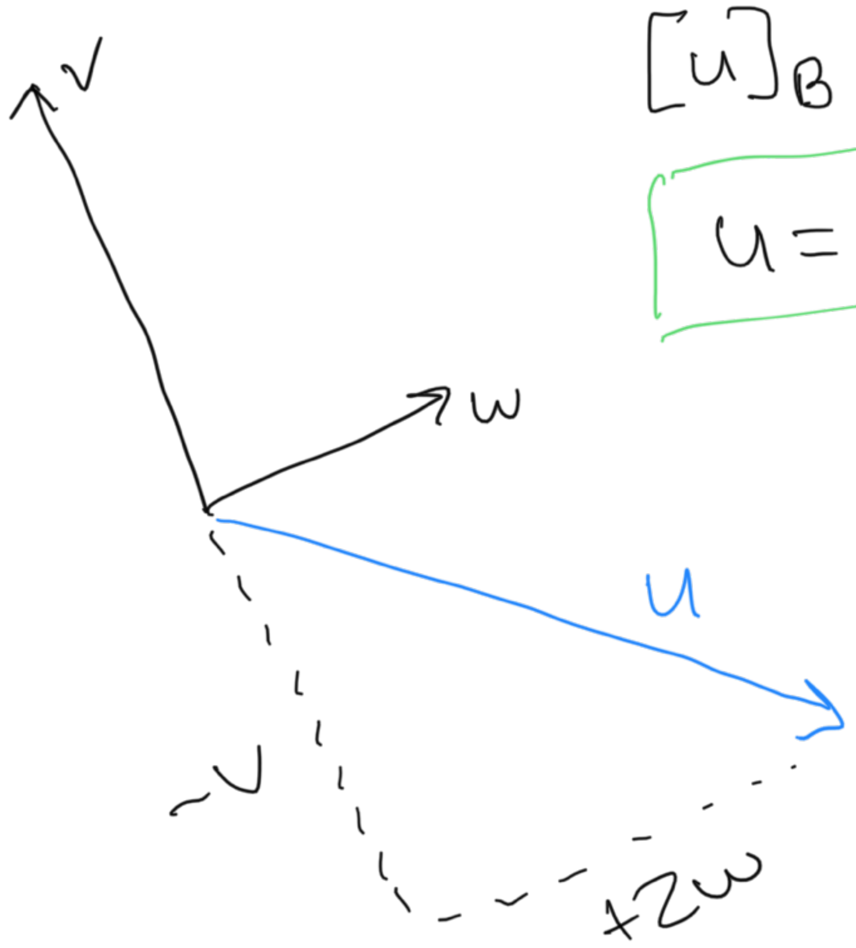
Homework 5 Solutions

1)

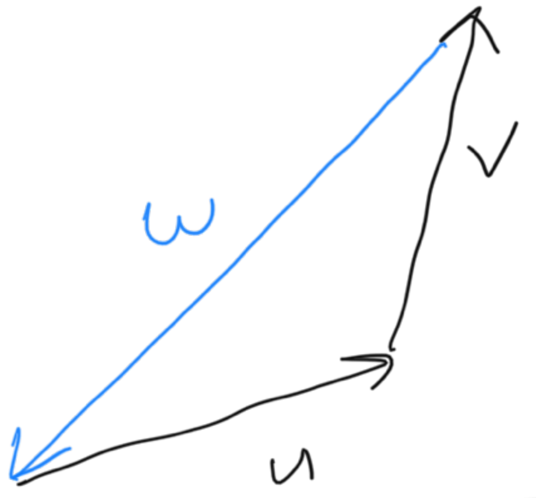


$$[u]_B = \begin{bmatrix}-1\\2\end{bmatrix}$$

$$\boxed{u = -v + 2w}$$

---

2)



$$w = -(u+v)$$

$$\boxed{[w]_B = \begin{bmatrix}-1\\-1\end{bmatrix}}$$

---

3) $B = \{e_1, e_2, e_3, e_4\}$

$C = \{e_3, e_1, e_2, e_4\}$

a) To find $P_{C \leftarrow B}$, we need to write each basis vector in B in terms of the basis vectors in C:

$$e_1 = c_1 e_3 + c_2 e_1 + c_3 e_2 + c_4 e_4$$

$$e_1 = 0e_3 + 1e_1 + 0e_2 + 0e_4$$

$$\text{So } [e_1]_C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly,

$$[e_2]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad [e_3]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad [e_4]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

These form the columns of $P_{C\leftarrow B}$:

$$\boxed{P_{C\leftarrow B} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

b) To find $P_{B\leftarrow C}$, we need to write each basis vector of C in terms of the basis vectors of B:

$$e_3 = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4$$

$$e_3 = 0e_1 + 0e_2 + 1e_3 + 0e_4$$

$$\text{So } [e_3]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Similarly,

$$[e_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [e_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad [e_4] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix}0\end{bmatrix} \qquad \begin{bmatrix}0\end{bmatrix} \qquad \begin{bmatrix}1\end{bmatrix}$$

These form the columns of $P_{B\leftarrow C}$:

$$\boxed{P_{B\leftarrow C} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}$$

Note that $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$ are inverses of each other, as expected

$$\underbrace{\begin{bmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}_{P_{C\leftarrow B}} \cdot \underbrace{\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}_{P_{B\leftarrow C}} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

---

4) $V = \left\{ f : \mathbb{R} \to \mathbb{R} : f(x) = \dfrac{c_1x + c_2}{(x-1)(x-2)},\ c_1, c_2 \in \mathbb{R} \right\}$

a) If V is a subspace, V is

i) Closed under addition

Consider $f_1 = \dfrac{ax+b}{(x-1)(x-2)}$ and

$f_2 = \dfrac{cx+d}{(x-1)(x-2)}$ as elements of V.

$$f_1 + f_2 = \frac{ax+b}{(x-1)(x-2)} + \frac{cx+d}{(x-1)(x-2)}$$

$$= (a+c)x + (b+d)$$

$$= \frac{\quad}{(x-1)(x-2)}$$

which is in V.

(ii) closed under scalar mult.

consider $f = \frac{ax+b}{(x-1)(x-2)}$

and $k \in \mathbb{R}$.

$$k \cdot f = k\left(\frac{ax+b}{(x-1)(x-2)}\right)$$

$$= \frac{(ka)x+(kb)}{(x-1)(x-2)}$$

which is in V.

Then V is a subspace.

b) $B = \left\{ \frac{1}{(x-1)(x-2)}, \frac{x}{(x-1)(x-2)} \right\}$

$C = \left\{ \frac{1}{x-1}, \frac{1}{x-2} \right\}$

To find $P_{C \leftarrow B}$, we need to write each basis vector of B in terms of the basis vectors of C:

i) $$\frac{1}{(x-1)(x-2)} = c_1\left(\frac{1}{x-1}\right) + c_2\left(\frac{1}{x-2}\right)$$

By finding a common denominator:

$$1 = c_1(x-2) + c_2(x-1)$$

To solve for c_1, set $x=1$:

$$1 = c_1(1-2) + c_2\cancelto{0}{(1-1)}$$

$$1 = -c_1 \rightarrow \boxed{c_1 = -1}$$

To solve for c_2, set $x=2$:

$$1 = c_1\cancelto{0}{(2-2)} + c_2(2-1)$$

$$1 = c_2 \rightarrow \boxed{c_2 = 1}$$

So $\left[\dfrac{1}{(x-1)(x-2)}\right]_C = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

ii) $\dfrac{x}{(x-1)(x-2)} = c_1\left(\dfrac{1}{x-1}\right) + c_2\left(\dfrac{1}{x-2}\right)$

$$x = c_1(x-2) + c_2(x-1)$$

Setting $x=1$:

$$1 = c_1(1-2) + c_2\cancelto{0}{(1-1)}$$

$$1 = -1c_1 \rightarrow \boxed{c_1 = -1}$$

Setting $x=2$:

$$2 = c_1\cancelto{0}{(2-2)} + c_2(2-1)$$

$$2 = c_2 \rightarrow \boxed{c_2 = 2}$$

So $\left[\frac{x}{(x-1)(x-2)}\right]_C = \begin{bmatrix}-1\\2\end{bmatrix}$.

Then $\boxed{P_{B\leftarrow C} = \begin{bmatrix}-1 & -1\\1 & 2\end{bmatrix}}$.

c) It's more convenient to use basis C, so we should find the component vectors $[v]_C$ for each integrand using

$$[v]_C = P_{C\leftarrow B}\cdot[v]_B$$

i) $$\frac{2x+1}{(x-1)(x-2)} = 1\cdot\left(\frac{1}{(x-1)(x-2)}\right) + 2\cdot\left(\frac{x}{(x-1)(x-2)}\right)$$

$$[v]_C = \begin{bmatrix}-1 & -1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-3\\5\end{bmatrix}$$

So $$\frac{2x+1}{(x-1)(x-2)} = -3\left(\frac{1}{x-1}\right) + 5\left(\frac{1}{x-2}\right)$$

So $$\int\frac{2x+1}{(x-1)(x-2)}\,dx = \int\left(\frac{-3}{x-1} + \frac{5}{x-2}\right)dx$$

$$= \boxed{-3\ln|x-1| + 5\ln|x-2| + C}$$

ii) Similarly,

$$[v]_C = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -8 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

So $\displaystyle\int \frac{3x-8}{(x-1)(x-2)}\,dx = \int \left(\frac{5}{x-1} - \frac{2}{x-2}\right)dx$

$$= \boxed{5\ln|x-1| - 2\ln|x-2| + C}$$

---

5) a) $\begin{bmatrix} -9 \\ 1 \\ -8 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Form augmented matrix to solve for c_1, c_2, c_3:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -9 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & -8 \end{array}\right]$$

$$\underset{R_3 = R_3 - R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -9 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & -1 & 1 \end{array}\right] \underset{R_3 = R_3 + 2R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -9 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 3 \end{array}\right]$$

$$\left.\begin{aligned} c_1 + c_2 + 2c_3 &= -9 \\ c_2 &= 1 \\ c_3 &= -3 \end{aligned}\right\} \quad \begin{aligned} c_1 &= -4 \\ c_2 &= 1 \\ c_3 &= -3 \end{aligned}$$

So $\boxed{[v]_B = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}}$

b) $$\begin{bmatrix}15\\-18\\-30\end{bmatrix}=c_1\begin{bmatrix}5\\-3\\0\end{bmatrix}+c_2\begin{bmatrix}1\\0\\0\end{bmatrix}+c_3\begin{bmatrix}1\\0\\2\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}5&1&1&15\\-3&0&0&-18\\0&0&2&-30\end{array}\right]\underset{R_3=\frac{1}{2}R_3}{\overset{R_2=\frac{1}{3}R_2}{\sim}}\left[\begin{array}{ccc|c}5&1&1&15\\-1&0&0&-6\\0&0&1&-15\end{array}\right]$$

$$\underset{R_1=R_1+5R_2}{\sim}\left[\begin{array}{ccc|c}0&1&1&-15\\-1&0&0&-6\\0&0&1&-15\end{array}\right]\underset{P(1,2)}{\overset{R_2=-R_2}{\sim}}\left[\begin{array}{ccc|c}1&0&0&6\\0&1&1&-15\\0&0&1&-15\end{array}\right]$$

$$\left.\begin{aligned}c_1\qquad\quad&=6\\c_2+c_3&=-15\\c_3&=-15\end{aligned}\right\}\quad\begin{aligned}c_1&=6\\c_2&=0\\c_3&=-15\end{aligned}$$

So $\boxed{[v]_B=\begin{bmatrix}6\\0\\-15\end{bmatrix}}$

c) $$\begin{bmatrix}-3\\-2\\-1\\2\end{bmatrix}=c_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\1\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+c_4\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1&1&1&1&-3\\1&1&1&0&-2\\1&1&0&0&-1\\1&0&0&0&2\end{array}\right]\rightarrow\begin{aligned}c_1+c_2+c_3+c_4&=-3\\c_1+c_2+c_3&=-2\\c_1+c_2&=-1\\c_1&=2\end{aligned}$$

$$c_1 = 2 \longrightarrow c_2 = -1-2 = -3$$

$$c_3 = -2-2+3 = -1$$

$$c_4 = -3-2+3+1 = -1$$

So $\boxed{[v]_B = \begin{bmatrix} 2 \\ -3 \\ -1 \\ -1 \end{bmatrix}}$

---

6) $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

↑ ↗ columns w/ leading 1's

$A = \begin{bmatrix} 2 & 4 & 1 & 11 & 14 \\ 1 & 2 & 1 & 8 & 10 \\ 4 & 8 & 2 & 22 & 28 \end{bmatrix}$

↑ ↗ Corresponding columns in A form a basis for colspace(A)

$\boxed{\text{basis: } \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}}$

---

7) a) $\begin{bmatrix} 2 & -3 \\ 0 & 0 \\ -4 & 6 \\ 22 & -33 \end{bmatrix}$

← zero row

← ← scalar multiples of row 1 (will become zero rows w/ row red)

$$\text{rank}(A) + \text{nullity}(A) = n$$

$$1 + \text{nullity}(A) = 2$$

$$\boxed{\text{nullity}(A) = 1}$$

b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ ← Same as row 1, ← Same as row 3

$$\text{rank}(A) + \text{nullity}(A) = n$$

$$2 + \text{nullity}(A) = 3$$

$$\boxed{\text{nullity}(A) = 1}$$

c) $\begin{bmatrix} 0 & 0 & 0 & -2 \end{bmatrix}$

$$\text{rank}(A) + \text{nullity}(A) = n$$

$$1 + \text{nullity}(A) = 4$$

$$\boxed{\text{nullity}(A) = 3}$$

---

8) $V = \{ f : \mathbb{R} \to \mathbb{R} \mid f'' - f = 0 \}$

a) If a function f is in V, then it satisfies the diff eq.

i) e^x: $(e^x)'' - e^x = e^x - e^x = 0$ ✓

✓

ii) e^{-x}: $(e^{-x})'' - e = e - e = 0$ ✓

iii) $\sinh(x)$: $(\sinh(x))'' - \sinh(x) = 0$ ✓

iv) $\cosh(x)$: $(\cosh(x))'' - \cosh(x) = 0$ ✓

b) To show two functions are linearly independent, use the Wronskian (4.5.20)

i) $W[e^x, e^{-x}] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$

$= (e^x)(-e^{-x}) - (e^x)(e^{-x})$

$= -e^0 - e^0 = \boxed{-2}$

Since $W[e^x, e^{-x}] \neq 0$, e^x and e^{-x} are lin indep.

ii) $W[\sinh x, \cosh x] = \begin{vmatrix} \sinh x & \cosh x \\ \cosh x & \sinh x \end{vmatrix}$

$= \sinh^2(x) - \cosh^2(x) = \boxed{-1}$

Since $W[\sinh(x), \cosh(x)] \neq 0$, $\sinh(x)$ and $\cosh(x)$ are lin indep

c) ignore

d) If $u(x)$ is a solution to $u''(x) - u(x) = 0$, then $u \in V$. As such, we can express u as a linear combination of basis vectors of V. Choosing $C = \{e^x, e^{-x}\}$, we write

$$u(x) = c_1 e^x + c_2 e^{-x}$$

We can find a particular solution if we use the given initial conditions to find c_1, c_2.

$$u(x) = c_1 e^x + c_2 e^{-x}$$

$$u(0) = 5 \rightarrow \boxed{c_1 + c_2 = 5}$$

$$u'(0) = -2 \rightarrow \boxed{c_1 - c_2 = -2}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & -1 & -2 \end{array}\right]$$

$$\underset{R_2 = R_2 - R_1}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -2 & -7 \end{array}\right] \rightarrow \begin{array}{l} c_1 + c_2 = 5 \\ c_2 = \frac{7}{2} \end{array}$$

$$c_1 = 5 - \frac{7}{2} = \frac{3}{2}$$

So, $\boxed{u(x) = \frac{3}{2}e^{x} + \frac{7}{2}e^{-x}}$

---

a) $y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$

a) $E(t) = \frac{1}{2}m(y')^2 + \frac{1}{2}ky^2$

$$E'(t) = my' \cdot \underbrace{y''}_{\text{from chain rule}} + ky$$

$$E'(t) = my'\left[-\frac{c}{m}y' - \frac{k}{m}y\right] + ky$$

from diff eq

$$E'(t) = -c(y')^2 - ky + ky$$

$$E'(t) = -c\underbrace{(y')^2}_{\text{always pos}}$$

$c > 0$

So $E'(t) \leq 0$ ✓

b) If $y(0) = 0$, mass is at equilibrium position

If $y'(0) = 0$, mass is at rest.

So $E(0) = 0$. Since $E'(t) \leq 0$, no energy can be given to the system. Then the mass remains

the equilibrium pos

at rest in the equilibrium pos,

or mathematically: $y(t)=0$ ✓

for all $t \geq 0$

c) Since $u(t)$ and $v(t)$ satisfy the diffeq, so does any linear combination of $u(t)$ and $v(t)$.

Consider a new solution

$$p(t) = u(t) - v(t).$$

$$p(0) = u(0) - v(0) = 0$$

equivalent

$$p'(0) = u'(0) - v'(0) = 0$$

equivalent

From (b), if $p(0)=0$ and $p'(0)=0$,

then $p(t)=0$ for all $t \geq 0$.

Then $p(t) = u(t) - v(t) = 0$

Then $u(t) = v(t)$ for all $t \geq 0$. ✓

d) See solutions to HW04 Q1 a).