HOMEWORK 6 SOLUTIONS

MATH 240-003 SPRING 2020

Problem 1

a) Let $c, d \in \mathbb{R}$ and $f, g \in C^2([a,b])$. Then we have

$$\begin{aligned} T(cf+dg) &= (cf+dg)'' - 16(cf+dg) \\ &= c(f''-16f) + d(g''-16g) \\ &= cT(f) + dT(g) \end{aligned}$$

showing that T is linear

b) Let $a, c \in \mathbb{R}$ and $A, C \in M_{n\times n}$. Then

$$\begin{aligned} T(aA+cC) &= (aA+cC)B - B(aA+cC) \\ &= a(AB-BA) + c(CB-BC) \\ aT(A) + cT(C) \end{aligned}$$

c) You will see in problem 2b) that this map T is multiplication by a 2×2 matrix on the left, thus linear.

Problem 2

a) Using the standard bases, columns of the matrix representing T are the images of standard basis vector under T:

$$\begin{aligned} T(1,0,0) &= (-1,-1,3,0) \\ T(0,1,0) &= (0,0,0,0) \\ T(0,0,1) &= (1,0,2,0) \end{aligned}$$

so

$$[T] = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 3 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

b) By the same argument.

$$[T] = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Problem 3

a) You may use the definition or just observe that the matrix addition and scalar multiplication is entry-wise. Let $x, y \in \mathbb{R}$ and $A = (a_{ij}), B = (b_{ij}) \in V$. Then

$$\begin{aligned} T(xA+yB) &= T\begin{pmatrix} xa_{11}+yb_{11} & xa_{12}+yb_{12} \\ xa_{21}+yb_{21} & xa_{22}+yb_{22} \end{pmatrix} = x(a_{11}+a_{22}) + y(b_{11}+b_{22}) \\ &= xT(A) + yT(B) \end{aligned}$$

b) As in problem 2, look where each basis vector in $\mathcal{B}$ maps to. So we get

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

c) Rank of the matrix is 1. So by the rank-nullity theorem, its nullity is $4-1=3$.

Problem 4

Observe that such rotation maps the standard basis vectors in the following way:

$$T(e_1) = e_2$$
$$T(e_2) = e_3$$
$$T(e_3) = e_1$$

so we get

$$[T] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Problem 5

Geometrically, if A is a rotation by $2\pi/7$ matrix, then A^7 is a rotation by $7 \times 2\pi/7 = 2\pi$, which is equivalent to not rotation, that is, the identity matrix. So the following matrix satisfies the condition $A^7 = I_2$

$$A = \begin{pmatrix} \cos(2\pi/7) & -\sin(2\pi/7) \\ \sin(2\pi/7) & \cos(2\pi/7) \end{pmatrix}$$

Problem 6

As in problem 2, look at where e_1, e_2 map to
a) We see that $(1, 0)$ maps to $(\sqrt{2}/2, \sqrt{2}/2)$ and $(0, 1)$ maps to $(-\sqrt{2}/2, \sqrt{2}/2)$. So the matrix is

$$\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

The rest of this problem works the same way

b)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

c) This can be obtained by multiplying two matrices representing each operation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$$

d) This can be obtained by multiplying two matrices representing each operation

$$\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$$

e)

$$\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

f) Inverse to such matrices are obtained by "undoing" what we did. For example, the inverse to the matrix in part a) is rotation by 45 degrees in clockwise direction. So for example, the inverse to the matrix in part c) is

$$\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Problem 7

Same problem as problem 6.
a) x_1-axis is fixed, so $Ae_1 = e_1$. On the x_2x_3-plane, we have rotation by 60 degrees, so

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}$$

Similar to the first part.

$$\begin{pmatrix} 1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \end{pmatrix}$$

Notice that if we ignore the second row and the second column, it is the rotation matrix in $\mathbb{R}^2$ by 60 degrees.

Problem 8

Again, we compute $T(X)$ for each $X \in \mathcal{C}$ and $T(Y)$ for each Y in $\mathcal{B}$.

$$\begin{aligned}
T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1/3 & -2/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1/3 & -2/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1/3 & -2/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix} = -2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

showing that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly, we get

$$\begin{aligned}
T\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \\
T\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\
T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix} = -2\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

so

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$$

Indeed for any two bases $\mathcal{B}$ and $\mathcal{C}$, we have

$$[T]_{\mathcal{B}}^{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[T]_{\mathcal{C}}^{\mathcal{C}}P_{\mathcal{C}\leftarrow\mathcal{B}}$$

that is, $B = P^{-1}AP$.

Since $\mathcal{C}$ is a standard basis, we easily compute

$$P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

We compute easily (since P is lower triangular)

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

Please check yourself that $B = P^{-1}AP$.

Problem 9

This is false. Consider $T : \mathbb{R}^2 \to \mathbb{R}$ by $T(1,0) = 1$ and $T(0,1) = 2$. Clearly, $\{1, 2\}$ is not linearly independent.

Problem 10

a) Since each dog mates with another dog of the same type, we read off the subtable of the first, fourth and the last columns to get

$$M = \begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/4 & 1 \end{pmatrix}$$

b) Try this yourself. We will soon see that this is easily answered by diagonalizing M.