HOMEWORK 6 SOLUTIONS

MATH 240-003 SPRING 2020

1 Problem 1

Comment: V is actually a subset of functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}$.
a) It's straightforward to check that $T(af+bg) = (af(0)+bg(0), af'(0)+bg'(0))^T = aT(f)+bT(g)$.

b) Let $f \in \ker(T)$. This means $f(0) = f'(0) = 0$. From homework 5, since f satisfies the given ODE, $f(x) = 0$ for all $x \geq 0$. Hence, $\ker(T)$ consists of the zero vector, zero function only.

c) By rank nullity, we have $\text{rank}(T) + \text{nullity}(T) = \dim(V)$. However, part b) showed that the nullity is 0, and the rank is bounded by the dimension of $\mathbb{R}^2$, the codomain, so we have $\dim(V) = \text{rank}(T) + \text{nullity}(T) \leq \dim(\mathbb{R}^2) = 2$.

2 Problem 2

a) This is true. Eigenvalues are roots of the characteristic polynomial.

b) False as it may not be true. Consider

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Clearly e_1 is an eigenvector for the first but not for the second.

c) True. If $v, w \in E_\lambda$, then $A(bv+cw) = bAv + cAw = \lambda bv + \lambda cw = \lambda(bv+cw)$.

d) True. Any scalar multiple of an eigenvector is an eigenvector.

e) False. If v, w are eigenvectors for two different eigenvalues, their sum will not be an eigenvector of A.

f) False. Any identity matrix is a counterexample/

g) True if different means linearly independent. In this case, we're forced to have algebraic multiplicities match the geometric multiplicities.

h) True. Say $Av = \lambda v$. Then $A^5 v = \lambda A^4 v = \lambda^2 A^3 v = \cdots = \lambda^5 v$. So v is an eigenvector of A^5 with eigenvalue λ^5.

i) True. By h), A and A^2 share eigenvectors, and if $Av = \lambda v$, then $A^2 v = \lambda^2 v$. So eigenvalues of A^2 are squares of eigenvalues of A. So either 3 or -3 needs to be an eigenvalue of A.

j) True since $\det(A - xI) = \det((A-xI)^T) = \det(A^T - xI)$ and part a)

k) True. Consider the 0 matrix.

3 Problem 3

Notice that this matrix is uppertriangular, so taking $\det(A - xI)$, we get $(1-x)^2(4-x)^4(2-x)^2$, so eigenvalues are 1 with multiplicity 2, 4 with 4 and 2 with 2. Dimensions of eigenspaces are the nullity of the matrix $A - \lambda I$ for each $\lambda = 1, 4, 2$. By inspection we can conclude that the dimension of each eigenspace is $2, 2, 1$ respectively. Here's an example of why for the eigenvalue 4: if we look at $A - 4I$, notice that all diagonal entries are nonzero except for the block of 4's in the middle. So the nullity of the matrix $A - 4I$ is completely determined by the nullity on the 4×4 submatrix in the center:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is 2.

4 Problem 4

Simplest example should be diagonal, since the whole point of finding eigenvectors was in diagonalizing the matrix:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

5 Problem 5

Geometrically, can you come up with two linearly independent vectors that only gets scaled, either by a positive or negative factor when reflected across the line $y = x$? First one is $(1, 1)$ which lies above the line $y = x$ which is preserved under the reflection. Another is $(1, -1)$ which is perpendicular to $y = x$ and becomes $(-1, 1) = -1 \cdot (1, -1)$. So we conclude that $(1, 1)$ is an eigenvector with eigenvalue 1 and $(1, -1)$ is with -1.

6 Problem 6

a) First compute the eigenvalues:

$$\det \begin{pmatrix} 1-x & 6 \\ 2 & -3-x \end{pmatrix} = (1-x)(-3-x) - 12 = x^2 + 2x - 15 = (x+5)(x-3)$$

so eigenvalues are -5 and 3 each with multiplicity 1.
Let's find an eigenvector of -5 by finding a basis for the kernel of $A - (-5)I = A + 5I$:

$$\begin{pmatrix} 6 & 6 \\ 2 & 2 \end{pmatrix}$$

it's easy to find a vector in the kernel of this matrix: $(1, -1)^T$. **Tip:** If you're finding a single eigenvector of 2×2 matrix, Fix the first entry of your eigenvector or the second entry to be 1 and multiply out to find the remaining entry. This works because your eigenspace is a subspace.
Similarly, find the kernel of $A - 3I$:

$$\begin{pmatrix} -2 & 6 \\ 2 & -6 \end{pmatrix}$$

so $(3, 1)$ is an eigenvector for the eigenvalue 3.

b)Do the same:

$$\det \begin{pmatrix} 7-x & -8 & 6 \\ 8 & -9-x & 6 \\ 0 & 0 & -1-x \end{pmatrix} = (-1-x)((7-x)(-9-x) + 64) = -(x+1)^3$$

So -1 is the only eigenvalue with multiplicity 3. Let's find the eigenvalues by computing the kernel:

$$\begin{pmatrix} 8 & -8 & 6 \\ 8 & -8 & 6 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 8 & -8 & 6 \\ 8 & -8 & 6 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So any vector in the kernel is of the form $(y - 3/4z, y, z)$. So the kernel has a basis $(1, 1, 0)$ and $(-3/4, 0, 1)$: the dimension of this eigenspace is 2 and this matrix is not diagonalizable.

c) Same thing!!!!

$$\det \begin{pmatrix} 1-x & 1/4 & 0 \\ 0 & 1/2-x & 0 \\ 0 & 1/4 & 1-x \end{pmatrix} = (1-x)(1/2-x)(1-x)$$

so 1 is an eigenvalue with multiplicity 2 and 1/2 is with 1.
Compute the kernel of $A - 1I$:

$$\begin{pmatrix} 0 & 1/4 & 0 \\ 0 & -1/2 & 0 \\ 0 & 1/4 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has a basis $\{(1,0,0),(0,0,1)\}$. For 1/2:

$$\begin{pmatrix}1/2 & 1/4 & 0\\ 0 & 0 & 0\\ 0 & 1/4 & 1/2\end{pmatrix} \to \begin{pmatrix}1 & 1/2 & 0\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

whose kernel has a basis $\{(1,-2,1)\}$.

7 Problem 7

a) Same as problem 6. If you compute the determinant of $A - xI$, you get the charateristic polynomial $(x - \frac{3+\sqrt{17}}{2})(x - \frac{3-\sqrt{17}}{2})$. Because it's 2×2 matrix and each eigenspace has dimension 1 by the algebraic multiplicity, we can get the eigenvectors very easily. We'll start with a vector $(x,1)$ and find x so that this vector is in the kernel of $A - \frac{3+\sqrt{17}}{2}I$:

$$\begin{pmatrix}1-\frac{3+\sqrt{17}}{2} & -2\\ -2 & 2-\frac{3+\sqrt{17}}{2}\end{pmatrix}\begin{pmatrix}x\\1\end{pmatrix} = \begin{pmatrix}\frac{-1-\sqrt{17}}{2} & -2\\ -2 & \frac{1-\sqrt{17}}{2}\end{pmatrix}\begin{pmatrix}x\\1\end{pmatrix} = \begin{pmatrix}\frac{-1-\sqrt{17}}{2}x-2\\ -2x+\frac{1-\sqrt{17}}{2}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

so for the second entry to be 0, we must have $x = \frac{1-\sqrt{17}}{4}$. So $(\frac{1-\sqrt{17}}{4}, 1)$ is an eigenvector for the eigenvalue $\frac{3+\sqrt{17}}{2}$. Similarly we can find that $(\frac{1+\sqrt{17}}{4}, 1)$ is an eigenvector for the other eigenvalue.

So for

$$S = \begin{pmatrix}\frac{1-\sqrt{17}}{4} & \frac{1+\sqrt{17}}{4}\\ 1 & 1\end{pmatrix}$$

we have that $S^{-1}AS$ is diagonal with entries $\frac{3+\sqrt{17}}{2}$ and $\frac{3-\sqrt{17}}{2}$

Again, same problem.

$$\det(A - xI) = (x+4)(x-4)(x-1)$$

and the kernel of the matrix $A - \lambda I$ for each eigenvalue has dimension 1 with basis $(0,-1,1)^T$, $(0,7,1)^T$, and $(15,-7,2)^T$ respectively. So for

$$S = \begin{pmatrix}0 & 0 & 15\\ -1 & 7 & -7\\ 1 & 1 & 2\end{pmatrix}$$

we have that $S^{-1}AS$ is a diagonal matrix with the entries $4, -4, 1$.

8 Problem 8

We first observe that all rows are the same and so this matrix has nullity 4. But the nullspace of A is the eigenspace of A with eigenvalue 0, so 0-eigenspace has dimension 4. Next, notice that if v is any vector, then we have for any u,

$$\begin{pmatrix}-u-\\ -u-\\ -u-\\ -u-\\ -u-\end{pmatrix} v = \begin{pmatrix}u\cdot v\\ u\cdot v\\ u\cdot v\\ u\cdot v\\ u\cdot v\end{pmatrix} = u\cdot v\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$$

So if $Av = \lambda v$ for any v, v must be a multiple of $(1,1,1,1,1)^T$. Indeed, $(1,1,1,1,1)^T$ is an eigenvector with the eigenvalue $2020 \cdot 5 = 10100$. It's easy to find the kernel of A since it amounts to find a set of linearly independent set of vectors such that $(2020, 2020, 2020, 2020, 2020) \cdot v = (0,0,0,0,0)$. So for

$$\begin{pmatrix}1 & 0 & 0 & 0 & 1\\ -1 & 1 & 0 & 0 & 1\\ 0 & -1 & 1 & 0 & 1\\ 0 & 0 & -1 & 1 & 1\\ 0 & 0 & 0 & -1 & 1\end{pmatrix}$$

we have that

$$A = S\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 10100\end{pmatrix}S^{-1}$$

9 Problem 9

a) WE did this in problem 6c) technically:

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$$

So the middle diagonal matrix is the diagonalization of M. Call the matrix on the left S.

b)$M^n = (SDS^{-1})^n = SD^nS^{-1}$ as we showed in the earlier homework. So we get

$$M^n = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -2^{-n-1}(1-2^n) & 0 \\ 0 & 2^{-n} & 0 \\ 0 & -2^{-n-1}(1-2^n) & 2 \end{pmatrix}$$

c) so if we let $n \to \infty$, we can compute the limit by taking the limit in the middle matrix first.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}$$