HW08 Solutions

1) $x' = x + 4y$
$y' = 2x + 3y$ $\iff$ $\overset{(1)}{\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}}$

Change of variables: $\overset{(2)}{\begin{bmatrix} x \\ y \end{bmatrix} = S\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}$
(S is 2×2 mx)

Then $\begin{bmatrix} x' \\ y' \end{bmatrix} = S\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}$ but by (1),

$$A\begin{bmatrix} x \\ y \end{bmatrix} = S\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}$$ but by (2),

$$AS\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = S\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}$$ Then

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \underbrace{S^{-1}AS}_{D}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

want to make this diagonal

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ Diagonalize A:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

$\lambda_1 = 5$

$\lambda_2 = -1$

$\underline{\lambda_1 = 5}$

$$A - \lambda_1 I = A - 5I = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = -1}$

$$A - \lambda_2 I = A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

So, $\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$z_1' = 5z_1 \implies z_1 = c_1 e^{5t}$

$z_2' = -1z_2 \implies z_2 = c_2 e^{-t}$

$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} \\ c_2 e^{-t} \end{bmatrix}$. But $\begin{bmatrix} x \\ y \end{bmatrix} = S \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

So $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{5t} \\ c_2 e^{-t} \end{bmatrix}$, and

$$\boxed{\begin{aligned} x(t) &= c_1 e^{5t} + 2c_2 e^{-t} \\ y(t) &= c_1 e^{5t} - c_2 e^{-t} \end{aligned}}$$

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2) $e^{At} = S e^{Dt} S^{-1}$

a) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda) - 9 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda+2)(\lambda-4) \quad \lambda_1 = -2, \quad \lambda_2 = 4$$

$\underline{\lambda_1 = -2}$

$$A - \lambda I_1 = A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\underline{\lambda_2 = 4}$

$$A - \lambda_2 I = A - 4I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}}$$

b) $P(\lambda) = -(\lambda-2)^2(\lambda-1) = 0$

$\lambda_1 = 1$ w/ a.m. $= 1$

$\lambda_2 = 2$ w/ a.m $= 2$

g.m $= 1$

$\underline{\lambda_1 = 1}$

$A - \lambda_1 I = A - I = \begin{bmatrix} 5 & -2 & -1 \\ 8 & -3 & -2 \\ 4 & -2 & 0 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\underline{\lambda_2 = 2}$

$A - \lambda_2 I = A - 2I = \begin{bmatrix} 4 & -2 & -1 \\ 8 & -4 & -2 \\ 4 & -2 & -1 \end{bmatrix}$

g.m. $= 2$

$\left[\begin{array}{ccc|c} 4 & -2 & -1 & 0 \\ 8 & -4 & -2 & 0 \\ 4 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 4 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$v_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

$v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$$\boxed{e^{At} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 4 & 0 \end{bmatrix}^{-1}}$$

3) a) $\boxed{\text{False,}}$ if $\{y_1, y_2, \ldots, y_n\}$ is a set of sols to a diff eqn, then $W[y_1, y_2, \ldots, y_n] \neq 0$ everywhere for $\{y_1, y_2, \ldots, y_n\}$ to be lin indep.

b) $\boxed{\text{False}}$ (see Thm. 8.2.2)

c) $\boxed{\text{True,}}$ y_p solves $Ly = F$

u solves $Ly = 0$

$$L(y_p + u) = Ly_p + Lu = F + 0 = F.$$

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4) $L = (D-1)^2$ $\qquad y = xe^x$

$$Ly = (D-1)^2 xe^x = (D^2 - 2D + 1) xe^x$$

$$D^2(xe^x) = D(xe^x + e^x) = xe^x + 2e^x$$

$$D(xe^x) = xe^x + e^x$$

$$1(xe^x) = xe^x$$

$Ly = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x$

$\boxed{= 0.}$ So xe^x is in $\ker(L)$.

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5) $L = x^2 D + x$ $\qquad y = y(x)$

$Ly = x^2 y' + xy = 0$

$x^2 y' + xy = 0$ $\qquad$ Use integrating factor method (Ch. 1.6)

$y' + \frac{1}{x} y = 0$ $\qquad e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x$

Multiplying through by the I.F.:

$xy' + 1y = 0.$ This is now $\frac{d}{dx}(xy)$.

$\frac{d}{dx}(xy) = 0 \Rightarrow xy = c_1$

$$\boxed{y(x) = \frac{c_1}{x}}$$

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b) $y'' - 16y = 0$

For S to be a basis to this diff eq, S needs exactly 2 lin independent solutions,

✗ S_1 has only one sol

✗ S_2 has e^{2x}, which is not a sol

✗ S_3 has e^{2x}, which is not a sol

$\boxed{S_4}$ has 2 sols. Are they lin indep?

$$W[e^{4x}, e^{-4x}] = \begin{vmatrix} e^{4x} & e^{-4x} \\ 4e^{4x} & -4e^{-4x} \end{vmatrix} = -8$$

Since $-8 \neq 0$, e^{4x} and e^{-4x} are lin indep

✗ S_5 are not lin independent

$\boxed{S_6}$ has 2 sols. Are they lin indep?

$$W[\sinh(4x), \cosh(4x)] = \begin{bmatrix} \sinh(4x) & \cosh(4x) \\ 4\cosh(4x) & 4\sinh(4x) \end{bmatrix}$$

$$= 4(\sinh^2(4x) - \cosh^2(4x)) = -4$$

Since $-4 \neq 0$, $\sinh(4x), \cosh(4x)$ are lin indep

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7) Find 2 lin indep sols of the form $y(x) = e^{rx}$ to $y'' - 36y = 0$

$$y''(x) - 36y(x) = r^2 e^{rx} - 36e^{rx} = 0$$

$$r^2 - 36 = 0 \to r = \pm 6$$

Two sols are
$$\boxed{\begin{aligned} y_1(x) &= c_1 e^{6x} \\ y_2(x) &= c_2 e^{-6x} \end{aligned}}$$

Check linear independence:

$$W[e^{6x}, e^{-6x}] = \begin{vmatrix} e^{6x} & e^{-6x} \\ 6e^{6x} & -6e^{-6x} \end{vmatrix} = -12 \neq 0$$

lin indep.

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8) Find 3 lin indep sols of the form $y(x) = x^r$ to

$$x^3 y''' + x^2 y'' - 2xy' + 2y = 0, \quad x > 0$$

$$x^3 y'''(x) + x^2 y''(x) - 2xy'(x) + 2y(x)$$

$$= x^3\left(r(r-1)(r-2)x^{r-3}\right) + x^2\left(r(r-1)x^{r-2}\right)$$

$$-2x\left(r x^{r-1}\right) + 2x^r = 0$$

$$r(r-1)(r-2)x^r + r(r-1)x^r - 2rx^r + 2x^r = 0$$

$$r(r-1)(r-2) + r(r-1) - 2r + 2 = 0$$

$$r = -1, \ r = 1, \ r = 2$$

3 sols are
$$\boxed{\begin{aligned} y_1(x) &= c_1 x^{-1} \\ y_2(x) &= c_2 x \\ y_3(x) &= c_3 x^2 \end{aligned}}$$

Check lin independence:

$$W\left[\frac{1}{x}, x, x^2\right] = \begin{vmatrix} \frac{1}{x} & x & x^2 \\ \frac{-1}{x^2} & 1 & 2x \\ \frac{2}{x^3} & \frac{1}{x} & 2 \end{vmatrix}$$

$$= \frac{1}{x}(2-2) - x\left(-\frac{2}{x^2} - \frac{4}{x^2}\right) + x^2\left(\frac{-1}{x^3} - \frac{2}{x^3}\right)$$

$$= \frac{6}{x} - \frac{3}{x} = \frac{3}{x} \neq 0 \text{ for } x > 0.$$

Then y_1, y_2, y_3 are lin indep