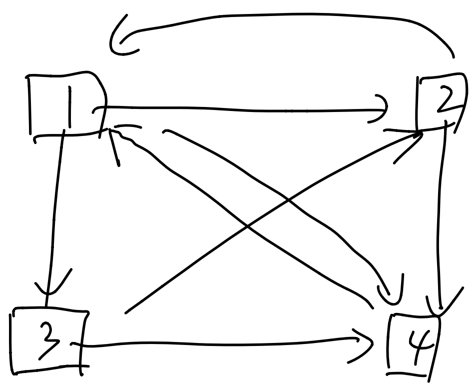
Linear algebra

1/21/2020

Google Page rank:

How to determine the importance of web pages by hyper links?

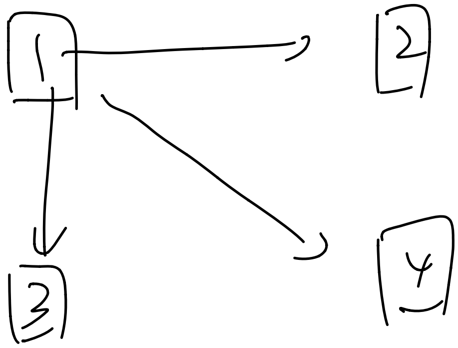


x_1 importance of page 1

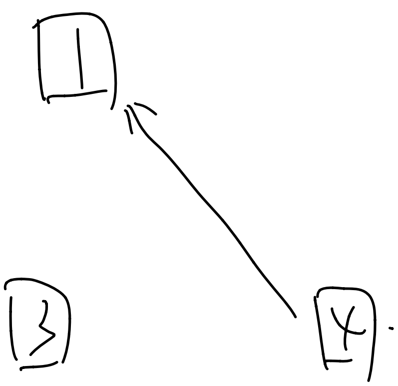
x_2 importance of page 2

x_3 importance of page 3

x_4 importance of page 4.



The importance of $\boxed{1}$, x_1 is transited to $\boxed{2}$ and $\boxed{3}$ equally.



$$x_1 = \frac{1}{2} x_2 + x_4.$$

$$x_2 = \frac{1}{3} x_1 + \frac{1}{2} x_3.$$

$$x_3 = \frac{1}{3} x_1$$

$$x_4 = \frac{1}{3} x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_3.$$

$$1\cdot x_1 - \frac{1}{2}x_2 + 0\cdot x_3 - 1\cdot x_4 = 0$$

$$-\frac{1}{3}x_1 + x_2 + \frac{1}{2}x_3 + 0\cdot x_4 = 0$$

$$-\frac{1}{3}x_1 + 0\cdot x_2 + x_3 + 0\cdot x_4 = 0$$

$$-\frac{1}{3}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + x_4 = 0$$

$$\begin{cases} x_1 - \frac{4}{3}x_4 = 0 \\ x_2 - \frac{2}{3}x_4 = 0 \\ x_3 - \frac{4}{9}x_4 = 0 \end{cases}$$

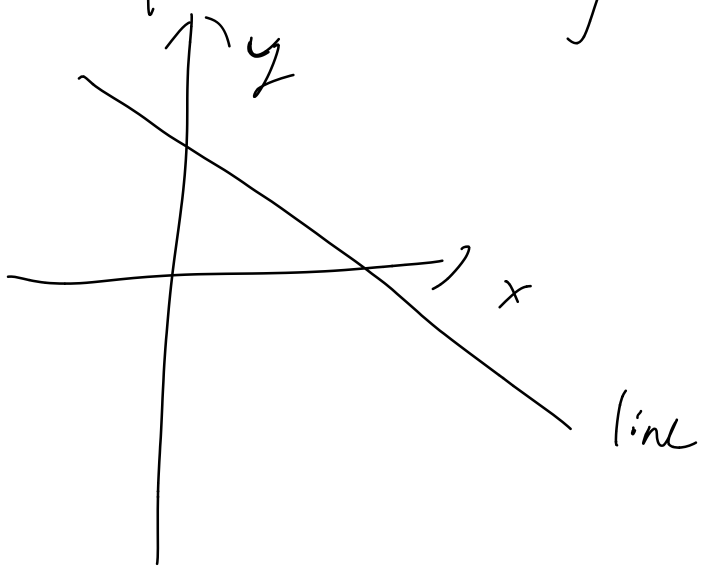
If $x_4 = 1$, then

$x_1 = \frac{4}{3}$, $x_2 = \frac{2}{3}$, $x_3 = \frac{4}{9}$, $x_4 = 1$

Linear algebra:

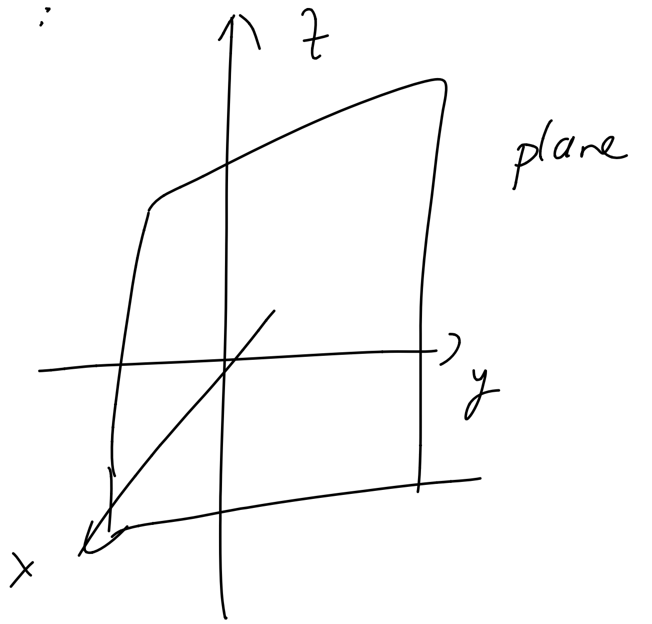
Solve systems of equations.

2D equation: $x+y=1$.



3D equation:

$x+y+z=1$.

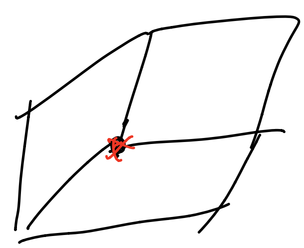


If (x, y, z) solves a system of equations, it lies on the intersection of those planes.

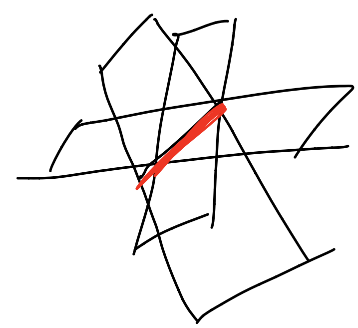
General fact:

A system of equations has either

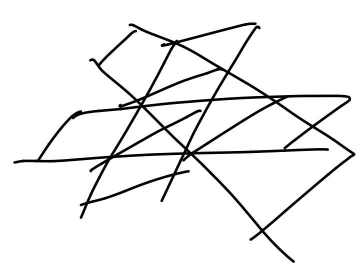
- A unique solution



- infinitely many solutions



- no solutions



How to determine which possibility and express all the solutions?

Ans: Row reduction / Gaussian or Gauss-Jordan elimination.

Ex:

$$x - 2y + 4z = 9$$
$$x - y + 2z = 5$$
$$3x - 5y + 13z = 32$$

$r_2 \mapsto r_2 - r_1$

$r_3 \mapsto r_3 - 3r_1$,

$$x - 2y + 4z = 9$$
$$y - 2z = -4$$
$$y + z = 5.$$

$$\xrightarrow{r_3 \mapsto r_3 - r_2} \quad \begin{aligned} x - 2y + 4z &= 9 \\ y - 2z &= -4 \\ 3z &= 9 \end{aligned}$$

$$\xrightarrow{r_3 \mapsto \frac{1}{3} r_3} \quad \begin{aligned} x - 2y + 4z &= 9 \\ y - 2z &= -4 \\ z &= 3 \end{aligned}$$

$$\xrightarrow{r_1 \mapsto r_1 + 2r_2} \quad \begin{aligned} x &= 1 \\ y - 2z &= -4 \\ z &= 3 \end{aligned}$$

$$\xrightarrow{r_2 \mapsto r_2 + 2r_3} \quad \begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Rmks on notation:

- $$\begin{aligned} x - 2y + 4z &= 9 \\ x - y + 2z &= 5 \\ 3x - 5y + 13z &= 32 \end{aligned} \quad \longleftrightarrow$$

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 2 \\ 3 & -5 & 13 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 32 \end{bmatrix}$$

$$A \cdot x = b$$

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 2 \\ 3 & -5 & 13 \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$b = \begin{bmatrix} 9 \\ 5 \\ 32 \end{bmatrix}$$

- Matrix vector multiplication:

  Matrix dot with column vector.

- column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$b = \begin{bmatrix} 9 \\ 5 \\ 32 \end{bmatrix}$$

row vector $r_1 = (1, -2, x)$

- Another interpretation of the same system:

$$\begin{aligned} x - 2y + 4z &= 9 \\ x - y + 2z &= 5 \quad \longleftrightarrow \\ 3x - 5y + 13z &= 32 \end{aligned}$$

$$x \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + y \cdot \begin{bmatrix} -2 \\ -1 \\ -5 \end{bmatrix} + z \cdot \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 5 \\ 32 \end{bmatrix}$$

Whether $\begin{bmatrix} 9 \\ 5 \\ 32 \end{bmatrix}$ is a "linear combination" of $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix}$.

A finial short hand.

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & 9 \\ 1 & -1 & 12 & 5 \\ 3 & -5 & 13 & 32 \end{array}\right]$$

"Augmented matrix".

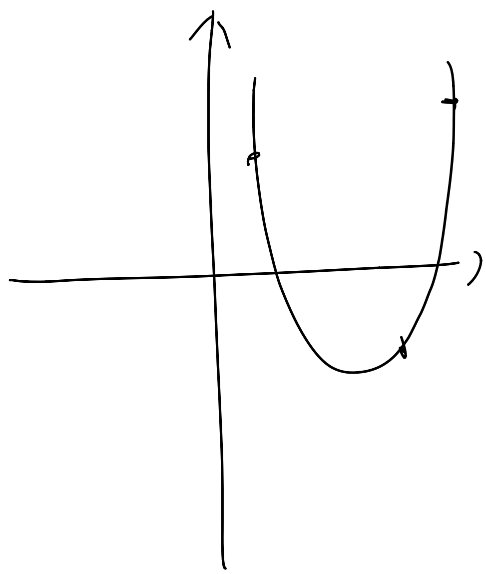
Elementary row operations:

(1) Mutiply any row by a non zero scalar

(2) Add to any row a scalar multiple of any other row

(3) Switch any two rows.

Is there a parabola passing through $(1,2)$, $(2,-5)$, $(3,4)$?



$y = ax^2 + bx + c$.

$$a + b + c = 2$$
$$4a + 2b + c = -5$$
$$9a + 3b + c = 4$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & -5 \\ 9 & 3 & 1 & 4 \end{bmatrix} \xrightarrow{\text{row reduce}}$$

$$\begin{bmatrix} 1 & & & 8 \\ & 1 & & -31 \\ & & 1 & 25 \end{bmatrix}$$

$$\Rightarrow \quad y = 8x^2 - 31x + 25.$$

An example with no solutions

$$a + b + c = 2$$
$$4a + 2b + c = -5 \quad \xrightarrow{\text{now reduce}}$$
$$5a + 3b + 2c = 0.$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad 0 = 3 \,!$$

contradiction !