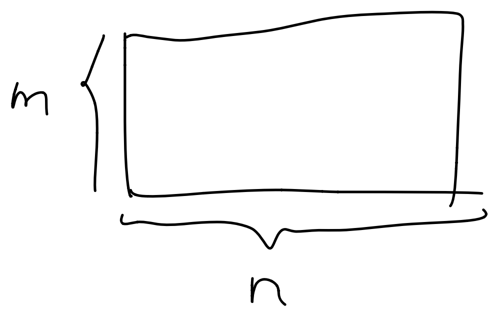
Rank-Nullity Theorem:

A $\quad m \times n$ matrix.



$$\underbrace{\text{rank } A}_{\substack{\dim \text{rowspace } A \subset \mathbb{R}^n \\ \dim \text{colspace } A \subset \mathbb{R}^m}} + \underbrace{\text{nullity } A}_{\dim \ker A \subset \mathbb{R}^n} = n.$$

Solutions to linear system.

Homogeneous. $\{x \mid Ax = 0\} = \ker A$.

General $Ax = b$.

Thm: $Ax = b$ has a solution if and only if $b \in \text{colspace } A$

Pf: $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

$$A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

linear combination

Thm: If $b \in$ colspace A, x_p is any particular solution.

solution set $\{x \mid Ax = b\} =$

$$\{x = x_p + y \mid Ay = 0\}.$$

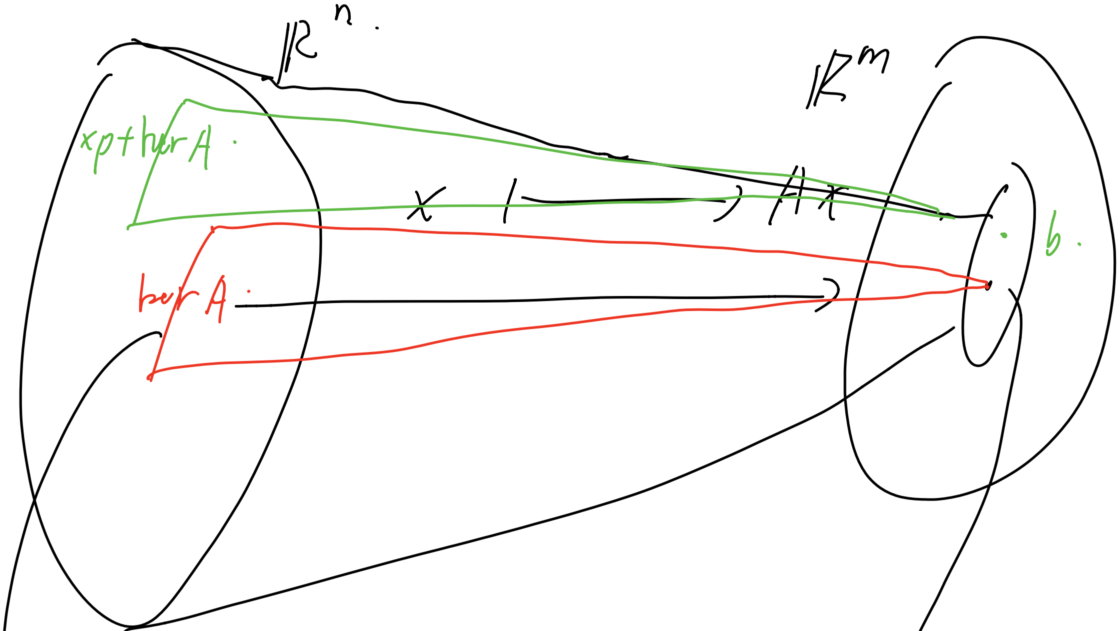
General solution = particular solution + solution to homogeneous equation.

Pf: If $x = x_p + y$, and $Ax_p = b$,
$Ay = 0$

Then $Ax = b$

If $Ax = b$, $Ax_p = b$. Let $y = x - x_p$.
then $Ay = 0$, and $x = x_p + y$,

Another point of view:



Colspace A.

= Image of T.

compressed to a point.

dim of $\mathbb{R}^n$

= dim of $\ker A$ (squeezed).

+ dim of Image T (left)

Linear transformation:

Defn: Given two vector spaces V, W.

A function (mapping)

$T: V \to W$ is called linear transformation

if a) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$.

b) $T(cv) = cT(v)$ for all $c \in \mathbb{R}$, $v \in V$.

Rmk: many functions are not linear:

$f(x) = \sin x$

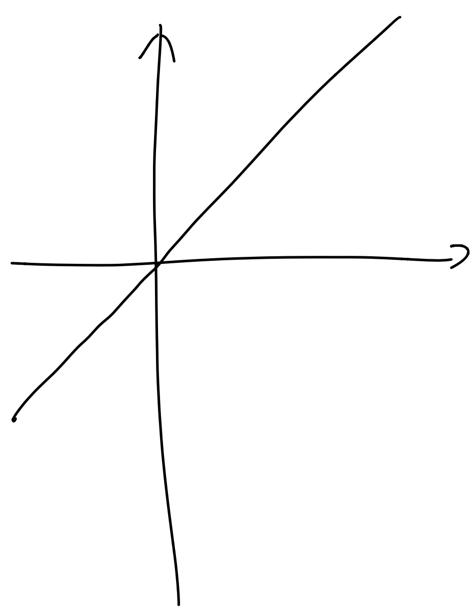
Ex: Linear transformations

$T : \mathbb{R} \to \mathbb{R}$.

Let $m = T(1)$.

Then $T(x) = T(x \cdot 1) = x \cdot T(1)$.

$= m \cdot x$



How about $\mathbb{R}^n \to V$.

Suppose $T: \mathbb{R}^3 \to V$ is a linear transformation

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = T\left(a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

$$= T\left(a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) + T\left(b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) + T\left(c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

$$= a\,T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \cdot T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \cdot T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

If $V = \mathbb{R}^m$,

then $\left[T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$

$= [T(e_1), \ T(e_2), T(e_3)] = A$

is a $m\times 3$ matrix and

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A\cdot\begin{pmatrix} a \\ b \\ c \end{pmatrix}. \quad \text{i.e.}$$

$$T(v) = A\cdot v.$$

Defn: $T: \mathbb{R}^n \to \mathbb{R}^m$,

$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, $\cdots$ $e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ is the standard basis,

The matrix of T is a $m\times n$ matrix $A = \begin{bmatrix} | & | & & | \\ T(e_1), & T(e_2), & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}$

Prop: $T(v) = A \cdot v$.

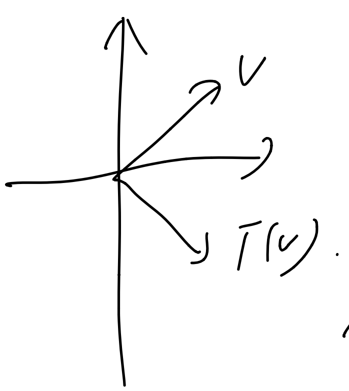
Fact: $T(0) = 0$ since

$T(0) = T(0+0) = T(0) + T(0)$

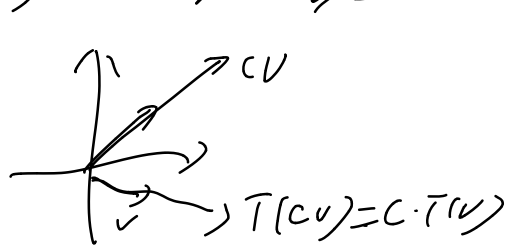
so $T(0) = 0$.

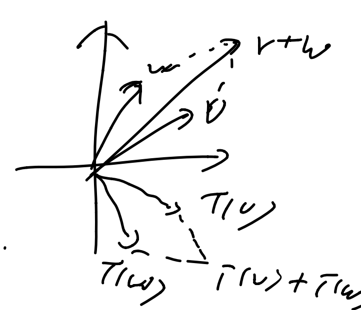
Some geometric examples: $\mathbb{R}^2 \to \mathbb{R}^2$.

① reflection with respect to x-axis.



T is a linear transformation because.





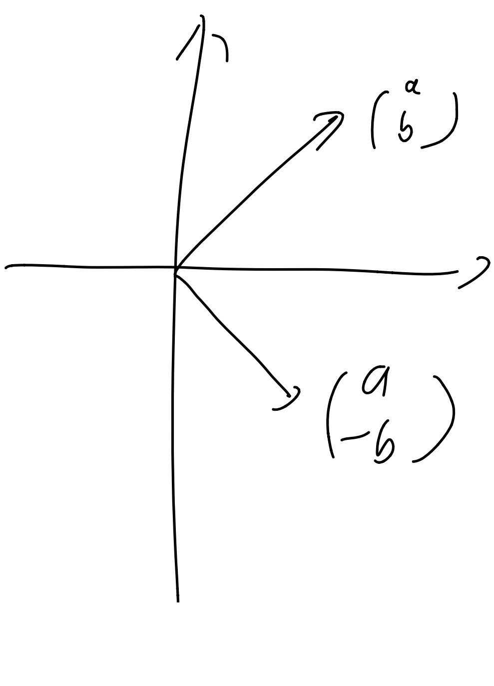
Matrix of T

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

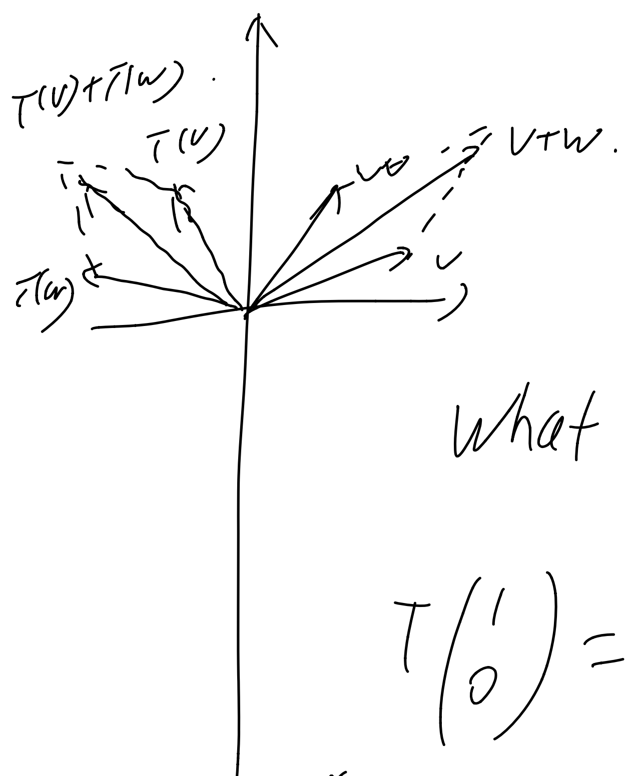
so $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$

$$T\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} a \\ -b \end{bmatrix}$$



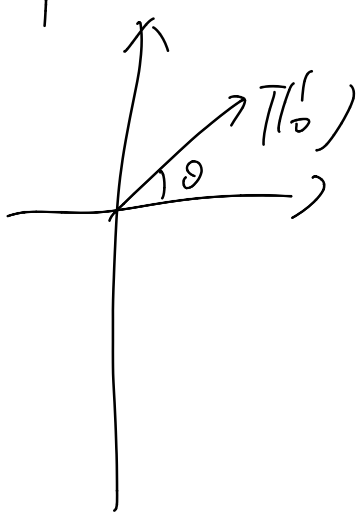
② rotation by θ counterclockwise.



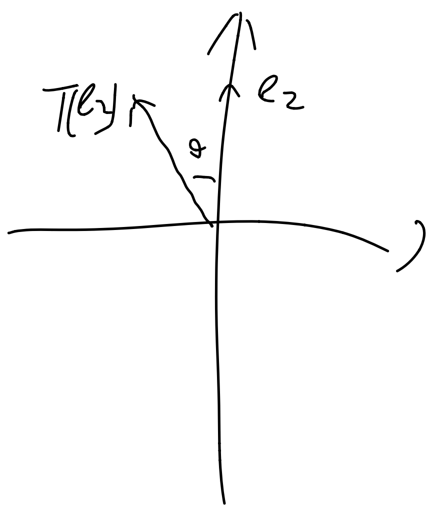
T is a linear transformation

What is the matrix.

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}.$$



$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$



So $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$.

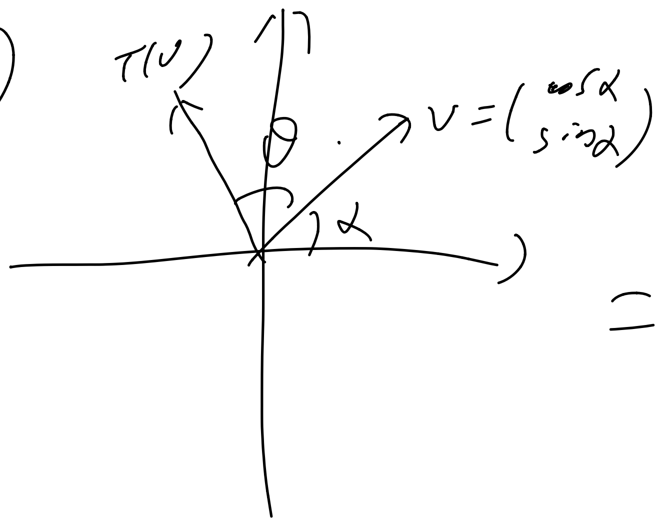
What is $T\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}$

Two ways to calculate:

a) $A\cdot\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\cdot\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}$

$$= \begin{bmatrix}\cos\theta\cos\alpha - \sin\theta\sin\alpha \\ \sin\theta\cos\alpha + \cos\theta\sin\alpha\end{bmatrix}$$

b)

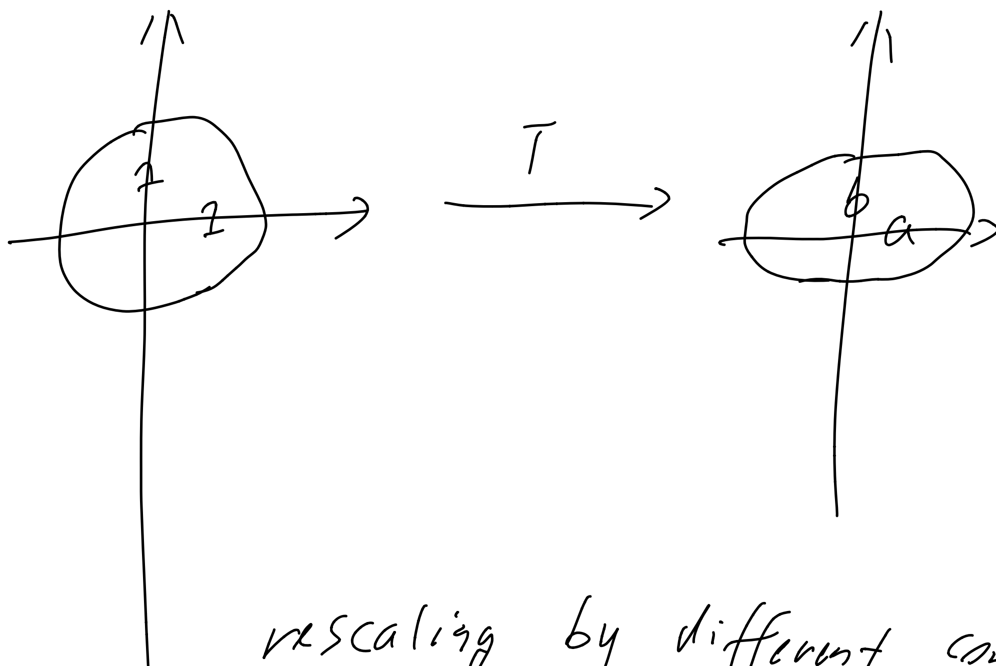


$T(v)$

$$= \begin{pmatrix}\cos(\theta+\alpha)\\ \sin(\theta+\alpha)\end{pmatrix}$$

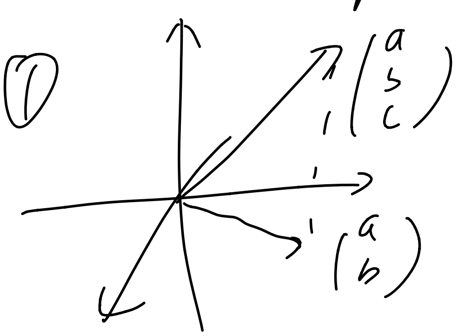
a) + b) ⇒ trig indentities!

③ $T(v) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot v.$



rescaling by different constants in different directions.

More examples

① 

projection:

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

matrix of T

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{so } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(2)

$T : \{\text{polynomial of deg} \leq 2\} \longrightarrow$

$\{\text{polynomial of deg} \leq 2\}$

$$T(f) = 2f - 3f' + f''.$$

T is a linear transformation.

(check this).