Lec 11.

Matrix for general linear transformation.

How to use matrix to study

$T: V \to W$ linear transformation.

Idea: $\mathbb{R}^n \to \mathbb{R}^m$.

we choose basis $\{e_1, \dots e_n\}$.

and use $[T(e_1), \dots T(e_n)]$

How to write everything in terms of

vectors in $\mathbb{R}^n$, $\mathbb{R}^m$?

Ans: Use component vectors.

Defn: Given two bases

$B = \{v_1 \ldots v_n\}$ for V

$C = \{w_1 \ldots w_m\}$ for W,

the matrix

$$[T]_B^C = \begin{bmatrix} | & | & & | \\ [T(v_1)]_C, & [T(v_2)]_C & \cdots & [T(v_n)]_C \\ | & | & & | \end{bmatrix}$$

is called the matrix representation of T relative to the bases B and C.

Prop: $[T(v)]_C = [T]_B^C \cdot [v]_B$

Pf: $T(v) = T\big((v_1, \ldots, v_n) \cdot [v]_B\big)$

$$= (T(v_1), \dots T(v_n)) \cdot [v]_{\mathcal{B}}$$

$$= (w_1, \dots w_m) \cdot [T]_{\mathcal{B}}^{C} \cdot [v]_{\mathcal{B}}.$$

Ex from last time:

$T : \{\text{polynomial of deg} \leq 2\} \longrightarrow$

$\{\text{polynomial of deg} \leq 2\}$

$T(f) = 2f - 3f' + f''.$

Choose $\mathcal{B} = C = \{1, x, x^2\}.$

$T(1) = 2, \quad [T(1)]_C = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$T(x) = 2x - 3, \quad [T(x)]_C = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

$T(x^2) = 2x^2 - 6x + 3$, $\quad [T(x^2)]_C = \begin{bmatrix} 2 \\ -6 \\ 2 \end{bmatrix}$.

So $[T]_B^C = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -3 & -6 \\ 0 & 0 & 2 \end{bmatrix}$

compute $T(4-6x+3x^2)$

$$[T(v)]_B = [T]_B^B [v]_B$$

$$= \begin{bmatrix} 2 & -3 & 2 \\ 0 & 2 & -6 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ -30 \\ 6 \end{bmatrix}$$

$T(v) = 32 - 30x + 6x^2$.

check by $2(4-6x+3x^2) - 3(4-6x+3x^2)' + (4-6x+3x^2)''$

Ex: Consider $T: M_{2\times 2} \to M_{2\times 2}$ by

$$T(M) = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \cdot M - M \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$

compute $[T]_B^B$ where

$$B = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}.$$

$$T(v_1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T(v_2) = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}$$

$$T(v_3) = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \quad T(v_4) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so $[T]_B^B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Change of basis.

B_1, B_2 are bases of V,

C_1, C_2 are bases of W.

Prop: $[T]_{B_2}^{C_2} = P_{C_2 \leftarrow C_1} [T]_{B_1}^{C_1} P_{B_1 \leftarrow B_2}$

The flow is from right to left.

Composition of two linear transformations.

$T_1 : V_1 \to V_2$

$T_2 : V_2 \to V_3$

$T_2 \circ T_1 : V_1 \longrightarrow V_3$.

$T_2 \circ T_1 (v) = T_2(T_1(v))$

Choose bases B_1 for V_1

B_2 for V_2

B_3 for V_3.

Prop: $[T_2 \circ T_1]_{B_1}^{B_3} = [T_2]_{B_2}^{B_3} [T_1]_{B_1}^{B_2}$.

Ex: compute $\begin{bmatrix} 17 & -6 \\ 35 & -12 \end{bmatrix}^{2020}$

using $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$

$= \begin{bmatrix} 17 & -6 \\ 35 & -12 \end{bmatrix}$

$$\underline{P_{B \leftarrow C}} \cdot [T]_C^C \, \underline{P_{C \leftarrow B}} = [T]_B^B$$

inverse to each other.

In fact $\begin{bmatrix} 2 & \\ & 3 \end{bmatrix}^{2020}$

and $\begin{bmatrix} 17 & -6 \\ 35 & -12 \end{bmatrix}$ is just matrix of the same $T^{2020} =$

$\underbrace{T \cdot T \cdot \cdots \cdot T}_{2020}$ under different

bases.