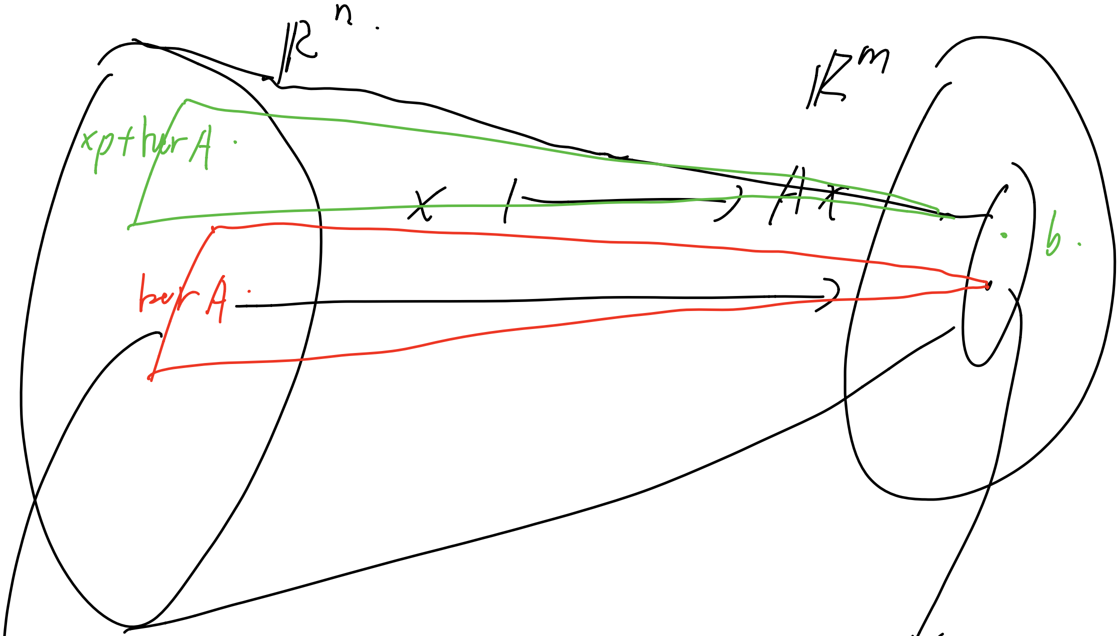
Recall rank-nullity theorem: $\mathbb{R}^n \to \mathbb{R}^m$.



compressed to a point.

Colspace A.

= Image of T.

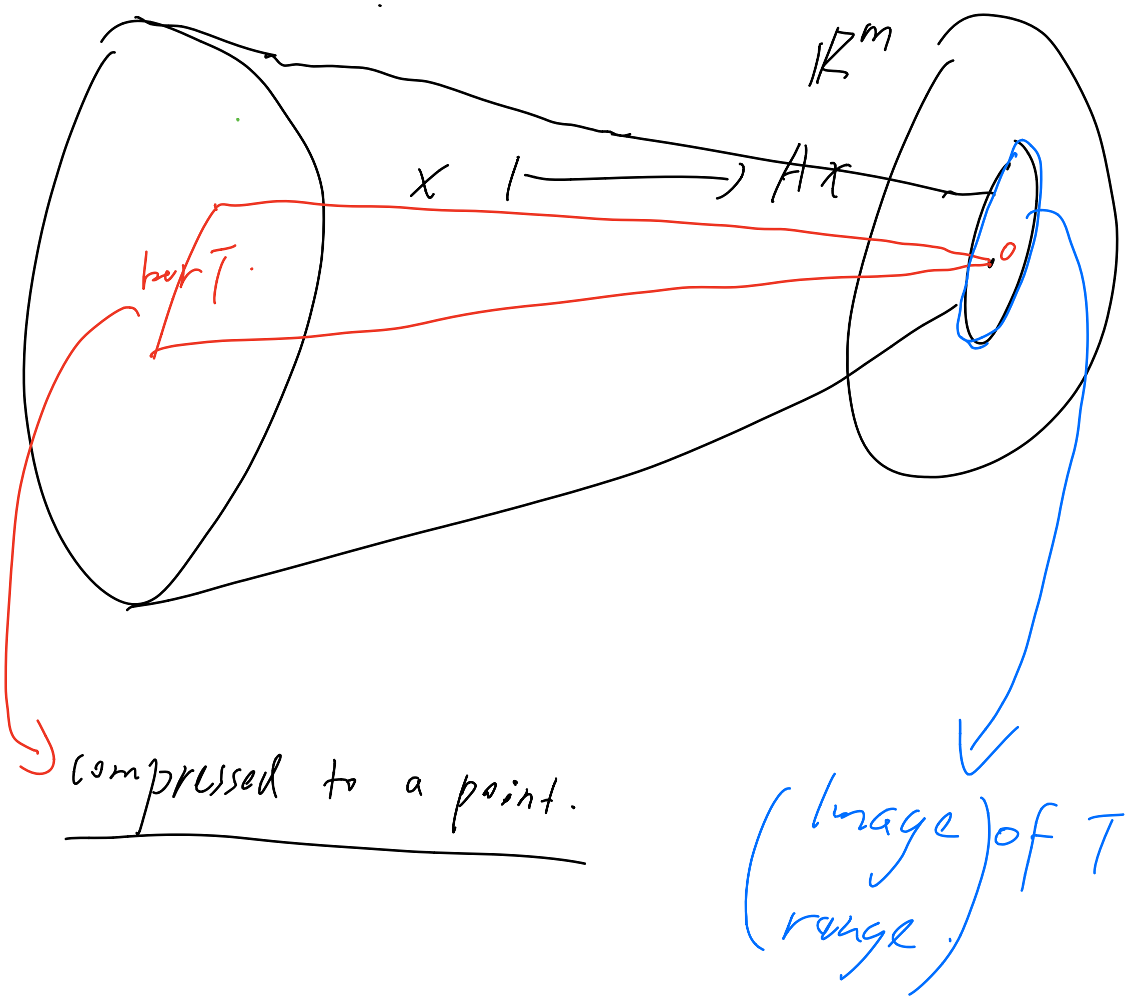
dim of $\mathbb{R}^n$

= dim of ker A (squeezed).

+ dim of Image T (left)

Rank-Nullity theorem for general linear transformation.

$T : V \to W$.



Defn: $\ker T = \{ v \in V \mid T(v) = 0 \}$

Defn: range of T $\quad$ Range T

image $\quad$ Image T

$$= \{ T(v) \mid v \in V \}.$$

Check $\ker T$ is subspace of V

Image T is subspace of W.

(closed under addition and

scalar multiplication

$0 \in \ker T$, $0 \in \text{Image } T$.)

Rank-Nullity Thm:

$$\dim \ker T + \dim \operatorname{Image} T = \dim V.$$

One application:

How to determine a polynomial $p(x)$ of $\deg \leq 3$ by taking special values?

x_1, x_2, x_3, x_4 distinct real numbers.

Prop: For any four points on $\mathbb{R}^2$,

$$(x_1, y_1), \ (x_2, y_2), \ (x_3, y_3), \ (x_4, y_4)$$

there exists a unique polynomial of degree ≤ 3 such that

$$y_i = p(x_i), \quad i = 1, 2, 3, 4.$$

Pf: Construct a linear transformation.

from: $V = \{\text{polynomials of deg} \le 3\}$

to $W = \mathbb{R}^4$.

by $T: V \to \mathbb{R}^4$

$$p(x) \mapsto T(p(x)) = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \end{pmatrix}$$

Let's look at $\dim V = 4$.

What is $\ker T$?

$$\ker T = \{ p(x) \in V \mid p(x_1) = p(x_2) = p(x_3) = p(x_4) = 0 \}$$

$$= \{ p(x) \in V \mid p(x) \text{ is a multiple of}$$

$(x-x_1)(x-x_2)(x-x_3)(x-x_4)\}$

$= \{p(x)=0\} = \{0\}.$

$\dim \ker T = 0.$

$\dim \text{Image } T = 4$ (rank-nullity)

So Image $T = \mathbb{R}^4$

T is surjective, which means

every $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ can be achieved by

some $T(p(x)) = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \end{pmatrix}$

Uniqueness: If $T(p_1(x)) = T(p_2(x))$

then $T(p_1(x) - p_2(x)) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$

So $p_1(x) - p_2(x) = 0$ because

$\ker T = 0$

$\Rightarrow p_1(x) = p_2(x)$.

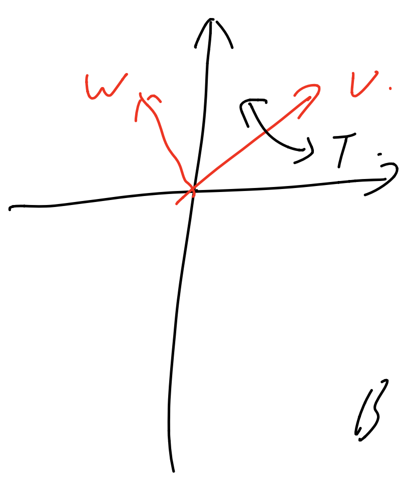
Chapter 7.

Eigenvalues and eigenvectors.

Motivation: Find "good basis" for

$T: V \to V$.

Ex: If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a reflection with respect to a line spanned by v.



Choose w to be a vector perpendicular to v. $w \perp v$

$\mathcal{B} = \{v, w\}$.

Then $T(v) = v$, $T(w) = -w$.

So $[T]_{B}^{B} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

Ex: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = [T]_B$

$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ Standard basis

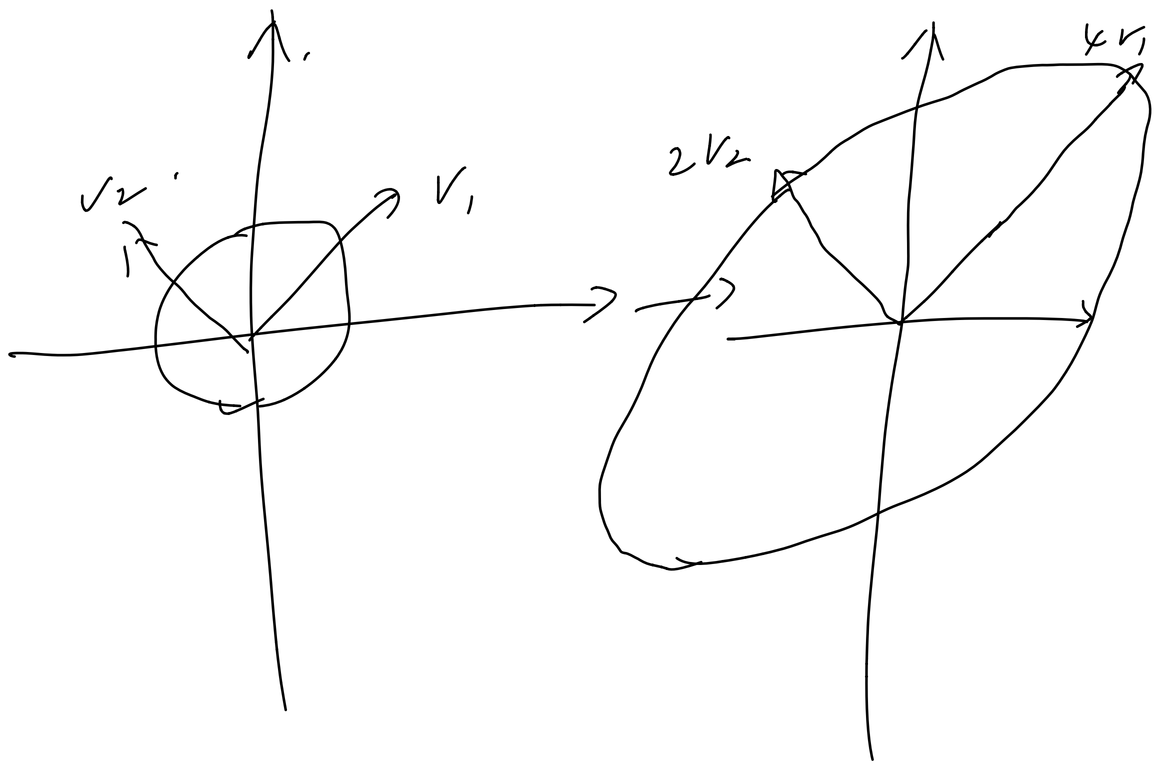
$T: \mathbb{R}^2 \to \mathbb{R}^2$.

Choose $C = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \right\}$.

Then $T(v_1) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4v_1$.

$$T(v_2) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2v_2.$$

$$\Rightarrow [T]_C = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$



$C = \{v_1, v_2\}$ is a better basis.

How to find "good basis"?

Defn: A nonzero vector $v \in V$ is an eigenvector of a linear transformation $T: V \to V$ if

$T(v) = \lambda v$ for some scalar λ.

The number λ is called the eigenvalue associated to v.

Note: The definition only makes sense when $T: V \to V$ the same vector spaces.

Making $[T]_{B}^{B}$ as diagonal as possible.
$\iff$ finding as many linear independent eigenvectors

as possible.

How to find eigenvectors and eigenvalues?

$T: \mathbb{R}^n \to \mathbb{R}^n$

T has matrix A.

$T(v) = Av$.

$T(v) = \lambda v \iff Av = \lambda v$

$(A - \lambda I)v = 0$ has a non-zero solution $v \neq 0$.

Then $\det(A - \lambda I) = 0$, i.e. $|A - \lambda I| = 0$.

Ex: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$,

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$$

$$= (\lambda - 2)(\lambda - 4).$$

So A has two eigenvalues

$\lambda_1 = 2$, $\lambda_2 = 4$.

How to find eigenvectors?

Solve $(A - \lambda I)v = 0$.

$\lambda_1 = 2$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v = 0$, $v_1 = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 4$, $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v = 0$, $v_2 = c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Choose $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ to be the good basis.