Eigenvalues are solutions of

$|A - \lambda I| = 0$. A $n \times n$ matrix.

(roots of)

Defn: $|A - \lambda I|$ is called the

<u>characteristic polynomial</u> of A.

(It is a degree n polynomial).

Defn: Eigenspace for a specific eigenvalue λ

is $\ker(A - \lambda I)$.

subspace of $\mathbb{R}^n$.

How to find eigenvalue and eigenspace?

① Compute $|A-\lambda I|$, find roots

② Solve $(A-\lambda I)v=0$, find bases of eigenspaces.

Ex: ① $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

$$|A-\lambda I| = (\lambda-1)^2.$$

eigenvalue $\lambda=1$,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

eigenspace $= \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \neq \mathbb{R}^2$.

② $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$|A - \lambda I| = \lambda^2 - 2\cos\theta\,\lambda + 1.$

If $\theta \neq 0, \pi$, no real eigenvalues.

Ex: $A = \begin{bmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & 24 & 13 \end{bmatrix}$, A has an eigenvalue $\lambda = 1$.

What is the corresponding eigenspace.

$A - I = \begin{bmatrix} 6 & -12 & 6 \\ 10 & -20 & 10 \\ 12 & -20 & 12 \end{bmatrix} \xrightarrow{\text{row reduce}}$

$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\ker(A-I) = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$\dim \ker(A-I) = 2.$$

Linear independence of eigenvectors

Prop: v_1, v_2 are eigenvectors corresponding to different eigenvalues λ_1, λ_2. then $\{v_1, v_2\}$ is linearly independent.

pf: $Av_1 = \lambda_1 v_1$

$Av_2 = \lambda_2 v_2$.

If $c_1 v_1 + c_2 v_2 = 0$, then

$A(c_1 v_1 + c_2 v_2) = 0$

$$c_1\lambda_1 v_1 + c_2\lambda_2 v_2 = 0$$

plug in $c_1 v_1 = -c_2 v_2$.

$$-\lambda_1 c_2 v_2 + c_2\lambda_2 v_2 = 0$$

$$(\lambda_2 - \lambda_1) c_2 v_2 = 0.$$

$$\lambda_2 - \lambda_1 \neq 0, \quad v_2 \neq 0$$

so $c_2 = 0$

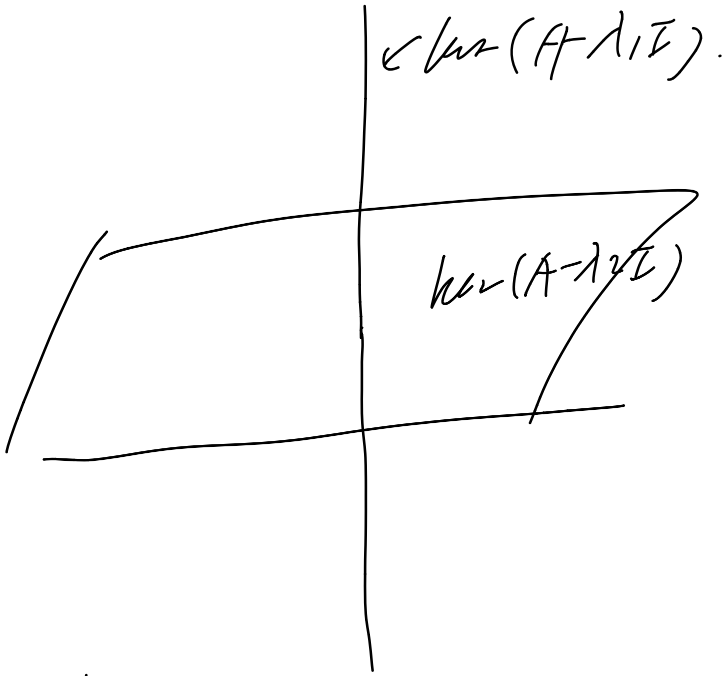
For the same reason $c_1 = 0$.

If $\{v_1 \cdots v_k\}$ is basis of eigenspace with eigenvalue λ_1.

$\{w_1 \cdots w_m\}$ is basis of eigenspace with eigenvalue λ_2

$\lambda_1 \neq \lambda_2$,

then $\{v_1, \cdots v_k, w_1 \cdots w_m\}$ is linearly independent.



$\ker(A-\lambda_2 I) \cap \ker(A-\lambda_1 I) = \{0\}$.

Defn: Say a $n\times n$ matrix A is diagonalizable if A has a basis of eigenvectors.

otherwise say A is defective

Ex: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ is diagonalizable because A has four distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 4$.

each has at least 1-dim'l eigenspace.

put all the eigenvectors together, we get a basis. since they're linearly independent.

Ex: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is diagonalizable.

$\dim \ker (A - I) = 1$

$\dim \ker (A - 2I) = 3$.

Ex: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

$\dim \ker(A - I) = 1.$

$\dim \ker(A - 2I) = 1$

$$A - 2I = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{rk} = 3$$

A defective.

Summary for algorithm to diagonalize A.

(1) Compute $|A - \lambda I| = 0$. find e-values $\lambda_1, \lambda_2 \dots$

(2) For each λ_i, find basis of $\ker(A - \lambda_i I)$.

(3) Check if sum of $\dim \ker(A-\lambda_i I)$

$= n$.

If it is $= n$,

use $S = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

eigenvectors.

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Defn: If A and B $n\times n$ matrices

and $S^{-1}AS = B$, then A, B

are called <u>similar</u>.

similar matrices have the same

characteristic polynomial and eigenvalues.

Fact: $|A-\lambda I| = (\lambda-\lambda_1)^{n_1} \cdots$

n_1 is called algebraic multiplicity.

$\dim \ker (A-\lambda_1 I) \leq n_1$.

Prop: $|A-\lambda I| = (\lambda_1-\lambda)\cdots(\lambda-\lambda_n)$

has n roots $\lambda_1 \cdots \lambda_n$

(not necessarily distinct).

① $\det A = \lambda_1 \cdots \lambda_n$

② $\operatorname{trace} A = \lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_n$.

Pf for A 2×2 matrix.

by computation of $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$

Ex: $A = \begin{bmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & 24 & 13 \end{bmatrix}$,

We already know $\lambda = 1$ is an eigenvalue.

$\dim \ker (A - I) = 2$

so A has at least $\lambda_1 = \lambda_2 = 1$.

$\lambda_1 + \lambda_2 + \lambda_3 = 7 - 19 + 13 = 1$.

so $\lambda_3 = -1$.