Lec 14

Eigenvector and eigenvalues.

A $n \times n$ matrix.

$$\underbrace{A v}_{\text{e-vector}} = \underbrace{\lambda}_{\text{e-value.}} v \qquad v \neq 0. \qquad v \in \mathbb{R}^n.$$

① λ is the root of $\underbrace{\det(A - \lambda I)}$

characteristic polynomial.

polynomial of λ with degree $= n$.

$(-\lambda)^n + \cdots\cdots + \cdots$

② Find eigenspace $\ker(A - \lambda I)$.

solve $(A - \lambda I)v = 0$. find basis

Diagonalization.

A is called diagonalizable if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors.

Otherwise A is called defective.

Ex:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 1: $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$

$$= (1-\lambda)(2-\lambda)(2-\lambda)$$

$$= (1-\lambda)(2-\lambda)^2 \Leftarrow$$

algebraic multiplicity of $\lambda_2 = 2$ is 2.

$\lambda_1 = 1, \quad \lambda_2 = 2.$

Step 2: Eigenspaces:

$\ker(A - I)$ $\quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\downarrow$

$\text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \quad \dim = 1.$

$\uparrow$ basis.

$\ker(A-2I)$ $\quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow rk = 2$

$\downarrow$

$\text{span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$ $\quad$ dim = 1.

$1 + 1 = 2 < 3$.

A is not diagonalizable.

defective.

Ex: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

step 1: $|A - \lambda I| = (1-\lambda)(2-\lambda)^2$.

$\lambda_1 = 1$. $\quad \lambda_2 = 2$.

Step 2: $\ker(A - I)$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{rk} = 2.$

$\downarrow$

$\text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$ $\dim = 1$

$\ker(A - 2I)$ $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{rk} = 1$

$\downarrow$

$\text{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$ $\dim = 2.$

$1 + 2 = 3$ A is diagonalizable.

basis consisting of e-vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Prop: algebraic multiplicity $\geq$ dim e-space for every e-value.

Thm: A is diagonalizable if and only if algebraic multiplicity $=$ dim e-space for each e-value

Cor: If A has n distinct e-values. then A is diagonalizable.

Pf: algebraic multiplicity $= 1$, $\geq$ dim e-space ≥ 1. //

Defn: Given A, B $n \times n$ matrices.

We say A and B are similar if there exists S invertible $n \times n$ matrix, such that

$$S^{-1} A S = B.$$

If A is diagonalizable, we choose

$$S = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

, $v_1 \dots v_n$ are e-vectors with e-value $\lambda_1 \dots \lambda_n$.

$(A v_i = \lambda_i v_i)$.

$$S^{-1} A S = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & 0 & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

e-vectors: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = S$.

$$\underline{S^{-1}AS} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\downarrow$

diagonalization of A.

Prop: If A is similar to B,

i.e. $S^{-1}AS = B$.

then ① $|A - \lambda I| = |B - \lambda I|$

② $\det A = \det B$

③ $\text{trace } A = \text{trace } B$.

Prop: $\det A$ = product of e-values

$\lambda_1 \lambda_2 \cdots \lambda_n$

Trace A = sum of e-values

$\lambda_1 + \lambda_2 + \cdots + \lambda_n.$

It is easy to see if A is

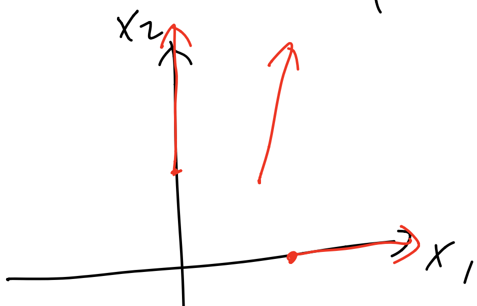
(diagonal or upp triangular matrix

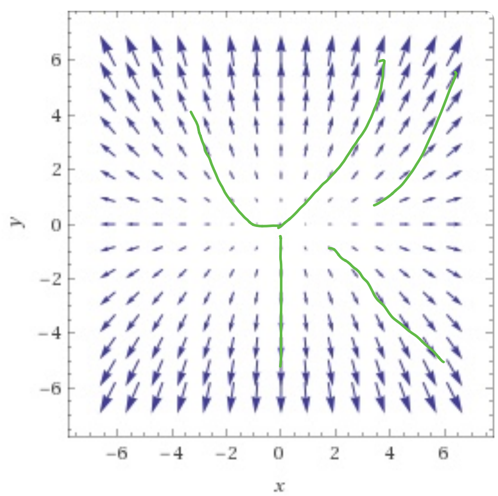
$$A = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

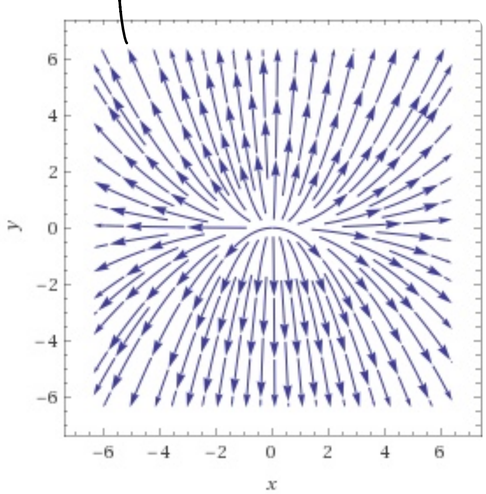
Application of diagonalization of ODE.

Integral curves (Flow lines) of vector fields.

$$\vec{F}(x_1, x_2) = \begin{pmatrix} 2x_1 \\ 4x_2 \end{pmatrix}.$$







flow lines

integral curves.

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$ velocity = vector field.
(flow)

$$\frac{dx(t)}{dt} = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 2x_1(t) \\ 4x_2(t) \end{pmatrix}$$

$$\underline{x'(t)} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underline{A \cdot x}.$$

$$x_1'(t) = 2x_1(t) \Rightarrow x_1(t) = C_1 e^{2t}$$

$$x_2'(t) = 4x_2(t) \Rightarrow x_2(t) = C_2 e^{4t}.$$

$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ is determined by $x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$

More general example:

$$\vec{F}(x_1, x_2) = \begin{pmatrix} 3x_1 + x_2 \\ x_1 + 3x_2 \end{pmatrix}$$

$$x'(t) = \begin{pmatrix} 3x_1 + x_2 \\ x_1 + 3x_2 \end{pmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$|A - \lambda I| = 0$, $\quad (3-\lambda)^2 - 1 = (\lambda-4)(\lambda-2)$

$\lambda_1 = 2 \xrightarrow{\text{solve e-vectors}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = v_1$

$\lambda_2 = 4 : \longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_2$.

$$S = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$S^{-1} A S = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Change of coordinates.

y is the coordinate under basis (v_1, v_2)

$$x = S \cdot y$$

$$x'(t) = A \cdot x(t).$$

$$(S \cdot y(t))' = A \cdot S \cdot y(t).$$

$$S\, y'(t) = A \cdot S \cdot y(t).$$

multiply S^{-1} on both sides

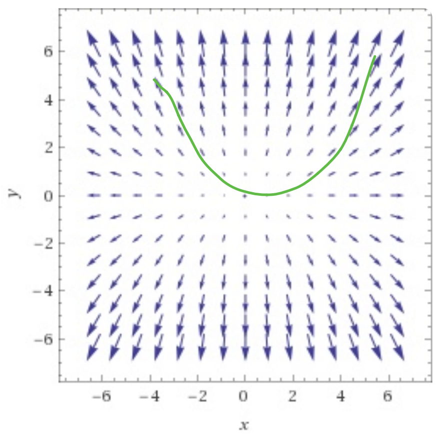
$$y'(t) = \underbrace{S^{-1} A S}_{= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}} \cdot y(t).$$

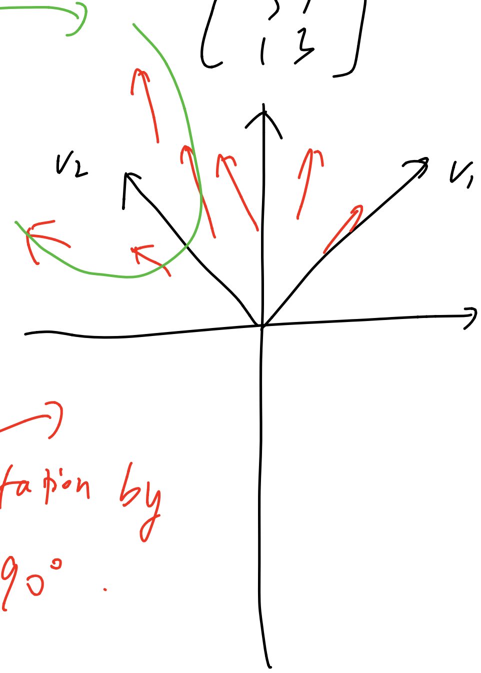
$$y(t) = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{4t} \end{pmatrix}$$

$$x(t) = S \cdot y(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{4t} \end{pmatrix}$$

$$= c_1 e^{2t} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \cdot e^{4t} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad\qquad \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$





rotation by 90°.

Solve ODE.

$$X'(t) = A \cdot X(t) \qquad A \quad n\times n.$$

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

by diagonalize A.

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Change of coordinate.

$$x(t) = S \cdot y(t).$$

$$y'(t) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \cdot y(t).$$

Solve $y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$

$$x(t) = S \cdot y(t).$$

(7.4) Matrix Exponential.

Recall $e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{3\times 2\times 1} + \frac{a^4}{4\times 3\times 2\times 1} + \frac{a^5}{5!} + \frac{a^6}{6!} + \cdots$

Exponential of A $n\times n$ matrix

Defn: $$e^A = I_n + A + \frac{A^2}{2} + \frac{A^3}{3\times 2\times 1} + \frac{A^4}{4!} + \cdots$$

$$e^{At} = I_n + At + \frac{A^2}{2}\cdot t^2 + \frac{A^3}{3\times 2\times 1} t^3 + \cdots$$

Prop: (I) If $AB = BA$,

$$e^{A+B} = e^A \cdot e^B$$

$$e^{(A+B)t} = e^{At} \cdot e^{Bt}$$

② $$(e^{A})^{-1} = e^{-A}$$

$$e^{At} = e^{-At}$$

Ex: $$A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$e^{At} = I_n + \begin{bmatrix} 2t & 0 \\ 0 & 4t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} (2t)^2 & 0 \\ 0 & (4t)^2 \end{bmatrix}$$

$$+ \frac{1}{3!} \cdot \begin{bmatrix} (2t)^3 & 0 \\ 0 & (4t)^3 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + 2t + \frac{(2t)^2}{2} + \cdots & \\ & 1 + 4t + \frac{(4t)^2}{2} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix}$$

For any $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & 0 \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

What if A is not diagonal?

If A is diagonalizable.

$$S^{-1} A S = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$e^{At} = I_n + At + \frac{(At)^2}{2} + \frac{(At)^3}{3!} + \cdots$$

$$= S \cdot S^{-1} + S D S^{-1} t +$$

$$\frac{1}{2!}A^2t^2 = \frac{1}{2!}SD^2S^{-1}t^2$$

$$(SDS^{-1})\cdot(SDS^{-1})$$

$$+\frac{1}{3!}SD^3S^{-1}t^3 + \cdots$$

$$= S\left(I_n + Dt + \frac{1}{2}D^2t^2 + \frac{1}{3!}D^3t^3 + \cdots\right)S^{-1}$$

$$= S\cdot \underline{e^{Dt}}\, S^{-1}$$

Ex: $A = \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$ $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$S^{-1}AS = \begin{bmatrix} 2 & \\ & x \end{bmatrix}$$

$$e^{At} = S \cdot (e^{Dt}) \cdot S^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} e^{2t} & \\ & e^{4t} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{2t} & \\ & e^{4t} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & e^{2t} \\ -e^{4t} & e^{4t} \end{bmatrix}$$

$$= \frac{1}{2} \begin{array}{c} \left[\begin{array}{c|c} e^{2t}+e^{4t}. & e^{2t}-e^{4t}. \\ e^{2t}-e^{4t} & e^{2t}+e^{4t} \end{array}\right] \\ \begin{array}{cc} \uparrow & \uparrow \\ v_1(t) & v_2(t) \end{array} \end{array}$$

$$\frac{d}{dt} e^{at} = a \cdot e^{at}$$

$$\frac{d}{dt} e^{At} = A \cdot e^{At}$$

related.

$$\frac{d}{dt} x(t) = A \cdot x(t).$$

$$e^{At} = \begin{bmatrix} | & | \\ v_1(t) & v_2(t) \\ | & | \end{bmatrix}$$

$$v_1'(t) = A \cdot v_1(t)$$

$$v_2'(t) = A \cdot v_2(t).$$

columns of e^{At} are <u>solutions</u> to $x'(t) = A \cdot x(t)$.

basis of solution space.

general solution $x(t) = c_1 v_1(t) + c_2 v_2(t)$