Chapter 8 and 9. ODE.

ODE ordinary differential equation

equation involving derivatives of a function.

$y(t)$ $y'(t)$, $y''(t)$, $y'''(t)$, $\dots$ $y^{(n)}(t)$

$$\frac{d}{dt}y(t), \quad \frac{d^2}{dt^2}y(t). \quad \frac{d^n}{dt^n}y(t).$$

$$D = \frac{d}{dt}, \quad y'(t) = Dy.$$

$$y''(t) = D^2y, \quad y^{(n)}(t) = D^n y.$$

Linear differential equations:

Ex: $y''(t) + 2y'(t) + y(t) = 0.$

$y''(t) + 2y'(t) + y(t) = \sin t.$

General form: $a_0(t) \cdot y^{(n)}(t) + a_1(t) \cdot y^{(n-1)}(t) + \dots$

$+ a_n(t) \cdot y(t) = F(t).$

divide $a_0(t)$ on both sides if $a_0(t) \neq 0$.

Goal : Solve this equation.

With initial condition

$$y^{(n-1)}(t_0) = C_1,\quad y^{(n-2)}(t_0) = C_2,\ \cdots\ y(t_0) = C_n.$$

This is called initial value problem.

Usually we focus on $a_0(t) = 1$.

(Defn) Linear differential operator of order n.

$$L = \underline{D^n + a_1(t)D^{n-1} + a_2(t)D^{n-2} + \cdots + a_n(t)}.$$

L has order n.

$$Ly = y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)\cdot y.$$

(Goal) Solve $Ly = F(t)$ (*)

when $a_1(t), a_2(t), \ldots a_n(t)$ are constants.

Want to study (*) from the point of view of linear algebra.

Compare (*) with $Ax = b$.

First view L as a linear transformation.

$V = \{$ functions of t with derivatives of all orders $\}$. (smooth functions)

L is a linear transformation from V to V

$$L = D^2 + 2D + 1.$$

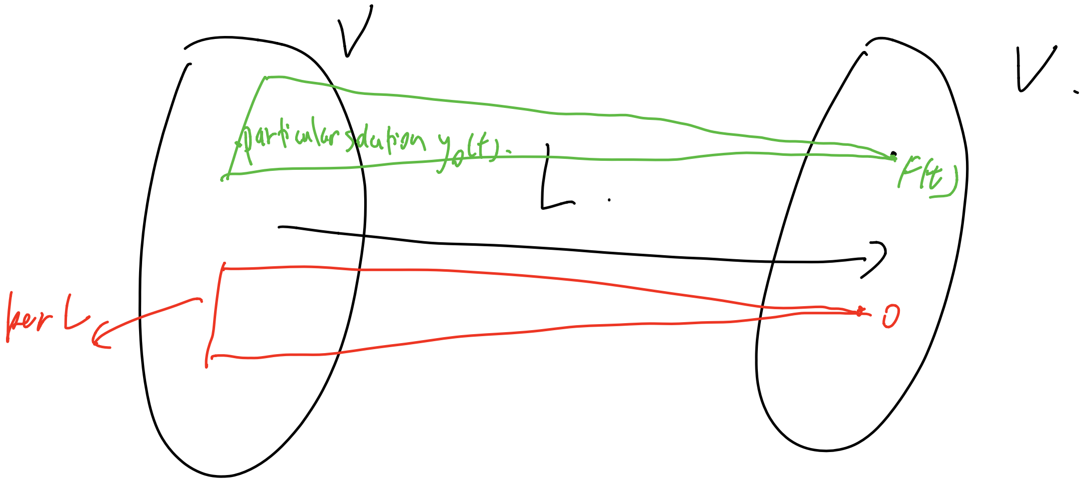
$$L : V \longrightarrow V.$$

$$y(t) \longmapsto L(y(t)) = y''(t) + 2y'(t) + y(t)$$

Check: L is a linear transformation.

$$L(y_1 + y_2) = Ly_1 + Ly_2.$$

$$L(c \cdot y) = c \cdot Ly.$$



$$L\,y(t) = F(t) \quad (*)$$

Thm: If $y_0(t)$ is a particular solution to (*).
then all the solutions to (*) has the form $y(t) = y_0(t) + x(t)$.

$x(t)$ is a solution to homogeneous equation $L[x(t)] = 0$

in ker L.

Two Steps

① Find ker L. Chapter 8.2

Solve homogeneous equation

$$Ly = 0.$$

② Find one particular solution $y_0(t)$. chapter 8.3
8.4.

Today: ①. homogeneous equation.

$$Ly = 0.$$

Ex: $L = D^2 + 2D + 1$. ,

$y''(t) + y'(t) + y(t) = 0$.

$L = D^2 + 2tD + 3t^2$

$\underline{y'' + 2ty' + 3t^2 y = 0}$

Black Box Theorem: (*) $Ly = F(t)$

(Existence + uniqueness).

(*) has a unique solution for any fixed initial conditions

$y^{(n-1)}(t_0) = C_1$. $y^{(n-2)}(t_0) = C_2$,

$\ldots$ $y(t_0) = C_n$.

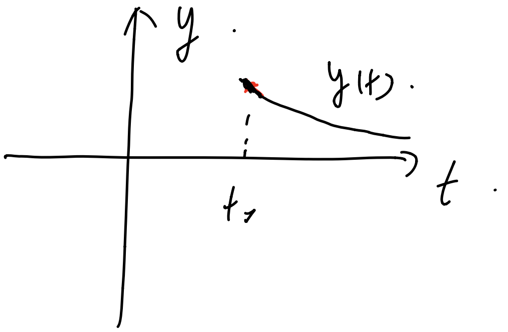
Ex of the theorem:

$y'(t) + 2y(t) = 0$

initial condition

$y(t_0) = 1$.

$\Rightarrow y'(t_0) = -2$.



From the black box thm:

Thm for $Ly = 0$.

L has order n.

$L = D^n + a_1(t) D^{n-1} + \cdots + a_n(t)$

(ker L is a subspace of V)

ker L has dim = n.

Thm tells us we should find

n linearly independent solutions to $Ly = 0$.

(basis of ker L)

all the solutions are linear combinations.

Ex: ① $L = D - 2t$.

$Ly = 0$.

$$\boxed{\begin{array}{c} D + a_1(t) \\ e^{\int a_1(t)} \end{array}}$$

$(D-2t)y = 0 \Rightarrow y' - 2ty = 0$.

(multiply both sides by $e^{\int -2t} = e^{-t^2}$.)

$$\underline{e^{-t^2}}(y' - 2t \cdot y) = 0.$$

$$e^{-t^2}y' + \underbrace{e^{-t^2} \cdot (-2t)}_{\begin{array}{c}\parallel \\ (e^{-t^2})'\end{array}} y = 0.$$

$$(e^{-t^2}y)' = 0.$$

$$e^{-t^2}y = C.$$

$$y = C \cdot e^{t^2}$$

ker L is 1-dim'l with basis $\{e^{t^2}\}$.

b) $L = D^2 - 4$

$Ly = y''(t) - 4y(t) = 0$.

$\boxed{\text{Assume } y(t) = e^{rt}}$ (*) <u>Guess</u>.

$Dy = re^{rt}$, $D^2y = r^2e^{rt}$

$r^2e^{rt} - 4e^{rt} = 0$. $\Rightarrow r^2 = 4$.

$r = \pm 2$. $y_1(t) = e^{2t}$

$y_2(t) = e^{-2t}$.

$\{y_1, y_2\}$ is linearly indepent (**).
hence basis of ker L.

any solution has the form

$y(t) = c_1e^{2t} + c_2e^{-2t}$.

Wronskian (Method to prove functions are linearly independent)

$y_1(t)$, $y_2(t)$.

$$\left\{\begin{array}{l} C_1 y_1(t) + C_2 y_2(t) = 0 \\ C_1 y_1' + C_2 y_2' = 0 \end{array}\right. \quad \swarrow \text{ derivative.}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\text{Wronskian} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(y_1, y_2)$$

If $W(y_1, y_2) \neq 0$, then $\{y_1, y_2\}$ is linearly independent. ↖ for some t.

Ex: $y_1(t) = e^{2t}$, $y_2 = e^{-2t}$

$$W(y_1, y_2) = \begin{vmatrix} e^{2t} & e^{-2t} \\ 2e^{2t} & -2e^{-2t} \end{vmatrix} = -2 - 2 = -4 \neq 0.$$

$y_1(t), \quad y_2(t), \quad y_3(t).$

$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0.$

derivative

$c_1 y_1' + c_2 y_2' + c_3 y_3' = 0$

derivative

$c_1 y_1'' + c_2 y_2'' + c_3 y_3'' = 0$

If $W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$ for

some t, then $\{y_1, y_2, y_3\}$ is linearly independent.

c). $L = D^2 + 4.$

$Ly = y'' + 4y = 0.$

$\boxed{\text{Assume } y(t) = e^{rt}}$

$$y'' = r^2 e^{rt}$$

$$r^2 e^{rt} + 4 e^{rt} = 0$$

$$\underline{r^2 + 4 = 0}. \quad r^2 = -4.$$

imaginary number.

$$r = \pm\sqrt{-4} = \pm 2\sqrt{-1} \qquad \boxed{i = \sqrt{-1}.}$$

Euler's formula.

$$\boxed{e^{\sqrt{-1}\theta} = \cos\theta + \sqrt{-1}\sin\theta.}$$

$$y_1(t) = e^{2\sqrt{-1}t} = \boxed{\cos 2t} + \sqrt{-1}\,\boxed{\sin 2t}$$

$$y_2(t) = e^{-2\sqrt{-1}t} = \cos(-2t) + \sqrt{-1}\sin(-2t)$$

$$= \boxed{\cos 2t} - \sqrt{-1}\,\boxed{\sin 2t}.$$

real solutions.

$$\begin{cases} \cos 2t = \dfrac{y_1 + y_2}{2} \\ \sin 2t = \dfrac{y_1 - y_2}{2\sqrt{-1}} \end{cases}$$

Next: $W(\cos 2t, \sin 2t) = \begin{vmatrix} \cos 2t & & \sin 2t \\ & \times & \\ -2\sin 2t & & 2\cos 2t \end{vmatrix}$

$$= 2(\cos 2t)^2 + 2(\sin 2t)^2 = 2 \neq 0$$

$\{\cos 2t, \sin 2t\}$ linearly independent.

hence basis of $\ker L$

general solution.

$$y(t) = C_1 \cos 2t + C_2 \sin 2t.$$

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① Prove dim $\ker L$ = order of L.

② L has constant coefficients,

how to find n linearly independent solutions

① Pf: We construct a linear transformation

from $\ker L \xrightarrow{T} \mathbb{R}^n$.

$$T: \underbrace{\ker L} \longrightarrow \mathbb{R}^n$$

$$y(t) \longmapsto T(y) = \begin{pmatrix} y(t_0) \\ y'(t_0) \\ y''(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{pmatrix}$$

Check T is a linear transformation.

rk - Nullity Thm :

$$\dim \ker L = \underline{\dim \ker \overset{\overset{0}{||}}{T}} + \underline{\dim \operatorname{Image} \overset{\overset{n\,.}{||}}{T}}.$$

① $\dim \underline{\ker T}$.

$$\ker T = \left\{ y \;\middle|\; \begin{array}{l} Ly = 0 \\ T(y) = 0 \end{array} \right\}$$

$$\swarrow$$

$$\begin{cases} Ly = 0 \\ y(t_0) = 0,\ y'(t_0) = 0, \ldots\ y^{(n-1)}(t_0) = 0. \end{cases}$$

$$\swarrow$$

black box theorem (uniqueness) $\Rightarrow y(t) = 0$.

$\ker T = \{0\}$. dim $= 0$

② $\dim \operatorname{Image} T$.

black box thm (Existence)

For any initial conditions

$$y(t_0),\ y'(t_0) \ldots\ y^{(n-1)}(t_0)$$

there exists $y(t)$, $L(y) = 0$.

$\Rightarrow \operatorname{Image} T = \mathbb{R}^n$, $\boxed{\dim = n.}$