Question 3 . A 4×4 matrix.

has e-values $\lambda_1 = 2$.

$\lambda_2 = -2$

$\lambda_3 = -2$

$\lambda_4 = 0$

① $Av = v$ has a solution v. True.

$v = 0$ is a solution.

② A is invertible False

$\boxed{\lambda_4 = 0}$ $\Rightarrow$ A is not invertible

$\det A = \lambda_1 \lambda_2 \lambda_3 \lambda_4 = 0$

A not invertible $\Leftrightarrow$ one of e-values is $= 0$.

③ $\det A = -2$ False.

$\det A = 0$

④ algebraic multiplicities are 1, 2, 1. True.

$\lambda_1 = 2$, $\lambda_2 = \lambda_3 = -2$, $\lambda_4 = 0$

⑤. A is diagonalizable (Non defective)

depends.

$$A = \begin{bmatrix} 2 & & & \\ & -2 & & \\ & & -2 & \\ & & & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & & & \\ & -2 & 1 & \\ & & -2 & \\ & & & 0 \end{bmatrix}$$ → increase the rank of $A+2I$.

not diagonalizable.

⑥ $Ax=0$ has infinitely many solutions

True.

$\lambda_4 = 0$ e-value.

⑦. $rk\,A = 4$ False.

$rk\,A = 4 -$ nullity of $A = 3$.

$\lambda_4 = 0$ algebraic multiplicity $\geq$ geometric multiplicity ≥ 1

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⑧ $\operatorname{tr} A = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2 - 2 - 2 + 0$

$= -2.$

⑨ nullity $A = 1$.

⑩ $\underline{A^2 v = 4v}$ has at least two linearly independent solutions.

True

$Av = \pm 2v$ $\quad \lambda_1 = 2 \quad v_1 \in$ e-space

$\lambda_2 = \lambda_3 = -2 \quad v_2 \in$ e-space.

$\{v_1, v_2\}$ linearly independent.

Question 4 $\quad A \quad n \times n$

$A^T A$ invertible $\iff$ A invertible.

Pf: $\det(A^T A) = (\det A^T)(\det A)$

$= (\det A)^2$

$\det(A^T A) \neq 0 \iff \det A \neq 0$.

ODE.

Outline:

① L linear differential operator. $y(x)$

$Ly = 0$. homogeneous.

solution space is a vector space.

dim = order of L = n

② L first order

$L = D + a_1(x)$

$$y' + a_1(x)y = 0$$

Integration factor $e^{\int a_1(x)}$ (multiply on both sides)

$$y' e^{\int a_1(x)} + a_1(x) \cdot y \cdot e^{\int a_1(x)} = 0$$

$$\left(y e^{\int a_1(x)}\right)' = 0 \quad \text{solve } y$$

(3) Wronskian of $y_1, y_2, \cdots y_n$.

$$\begin{vmatrix} y_1, & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W(y_1, y_2 \cdots y_n)$$

If $W \neq 0$ at some point, then $\{y_1 \cdots y_n\}$ linearly independent.

Solve $Ly = 0$ with constant coefficients.

Basic idea is to use $y(x) = e^{rx}$.

and determine r.

2^{nd}-order ODE.

$$y'' + a_1 y' + a_2 y = 0. \quad (*)$$

$a_1, a_2 \in \mathbb{R}$ constants.

Guess $y(x) = e^{rx}$ is a solution.

$$y' = r e^{rx}$$
$$y'' = r^2 e^{rx}$$

$(*) \Rightarrow \quad (r^2 + a_1 r + a_2) e^{rx} = 0$

$$\underline{r^2 + a_1 r + a_2 = 0.}$$

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Three different cases

Case 1. 2 real distinct roots. $a_1^2 - 4a_2 > 0$.

$$r_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}, \quad r_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}$$

$y_1(x) = e^{r_1 x}$. $\quad y_2 = e^{r_2 x}$

Need to check Wronskain.

$$W(y_1, y_2) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

$$= r_2 e^{(r_1 + r_2)x} - r_1 e^{(r_1 + r_2)x}$$

$$= (r_2 - r_1) e^{(r_1 + r_2)x} \neq 0$$

$(r_2 \neq r_1)$

All the solutions $y(x) = c_1 y_1 + c_2 y_2$.

$$= c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2: One repeated root. $a_1^2 - 4a_2 = 0$

$$r_1 = r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = -\frac{a_1}{2}.$$

Find one solution $y_1(x) = e^{rx}$. $r = -\frac{a_1}{2}$.

How to find the other solution

$$\underline{r^2 + a_1 r + a_2 = 0} \qquad \left(r + \frac{a_1}{2}\right)^2 = 0$$

$$(D^2 + a_1 D + a_2) = \left(D + \frac{a_1}{2}\right)^2$$

$$\left(D + \frac{a_1}{2}\right)\left[\underbrace{\left(D + \frac{a_1}{2}\right) y(x)}_{z(x)}\right] = 0$$

$$\left(D + \frac{a_1}{2}\right) z(x) = 0$$

$$z'(x) + \frac{a_1}{2} z(x) = 0 \qquad \text{multiply } e^{\int \frac{a_1}{2} dx}$$

$$= e^{\frac{a_1}{2}x}$$

$$\underbrace{z'(x) \cdot e^{\frac{a_1}{2}x} + \frac{a_1}{2} \cdot e^{\frac{a_1}{2}x} \cdot z(x)} = 0$$

$$\left(z(x) \cdot e^{\frac{a_1}{2}x}\right) = 0$$

$$z(x) \cdot e^{\frac{a_1}{2}x} = C_1$$

$$z(x) = C_1 e^{-\frac{a_1}{2}x}$$

$$\parallel$$

$$\left(D + \frac{a_1}{2}\right) y(x) = C_1 e^{-\frac{a_1}{2}x}$$

mutiply $e^{\frac{a_1}{2}x}$

$$y' e^{\frac{a_1}{2}x} + \frac{a_1}{2} e^{\frac{a_1}{2}x} y(x) = C_1$$

$$\left(y e^{\frac{a_1}{2}x}\right)' = C_1.$$

$$y e^{\frac{a_1}{2}x} = C_1 x + C_2.$$

$$y(x) = C_1 x \cdot e^{-\frac{a_1}{2}x} + C_2 e^{-\frac{a_1}{2}x}.$$

$$\begin{cases} y_1(x) = e^{-\frac{a_1}{2}x} = e^{r_1 x} \\ y_2(x) = x \cdot e^{-\frac{a_1}{2}x} = x \cdot e^{r_1 x} \end{cases} \qquad r_1 = r_2 = -\frac{a_1}{2}.$$

$$e^{rx}, \quad x e^{rx}$$

Case 3. $\underline{a_1^2 - 4a_2 < 0}$ two complex roots

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{-a_1 \pm \sqrt{4a_2 - a_1^2}\, i}{2}$$

$i^2 = -1$.

$r_1 = \alpha + \beta i, \quad r_2 = \alpha - \beta i.$

$\alpha = -\frac{a_1}{2}, \quad \beta = \frac{\sqrt{4a_2 - a_1^2}}{2}$

$y_1(x) = e^{r_1 x}, \quad y_2(x) = e^{r_2 x}$

Wronskian $(y_1, y_2) \neq 0$

Euler's identity:

$$y_1(x) = e^{(\alpha + \beta i)x} = e^{\alpha x} \cdot e^{\beta x i}$$

$$= e^{\alpha x} \cdot (\cos \beta x + i \sin \beta x)$$

$$y_2(x) = e^{\alpha x} \cdot e^{-\beta x \cdot i}$$

$$= e^{\alpha x} (\cos \beta x - i \sin \beta x)$$

$$\left.\begin{aligned} \frac{y_1+y_2}{2} &= e^{\alpha x}\cos\beta x \\ \frac{y_1-y_2}{2i} &= e^{\alpha x}\sin\beta x \end{aligned}\right\} \text{ basis of } \{Ly=0\}$$

Summary: if $L = D^2 + a_1 D + a_2$

- Case 1 $\quad a_1^2 - 4a_2 > 0$, $r^2 + a_1 r + a_2 = 0$ has two real distinct roots r_1, r_2

  $\{e^{r_1 x}, e^{r_2 x}\}$ is a basis of $\ker L$

- Case 2, $\quad a_1^2 - 4a_2 = 0$. $r^2 + a_1 r + a_2$

  $$= (r - r_1)^2$$

  $\{e^{r_1 x}, x e^{r_1 x}\}$ is a basis of $\ker L$

- Case 3, $\quad a_1^2 - 4a_2 < 0$, $r^2 + a_1 r + a_2$

  $$= (r - (\alpha + \beta i))(r - (\alpha - \beta i))$$

$\{e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x\}$ is a basis of $\ker L$.

Example: $y'' + 6y' + 25 = 0$.

$r^2 + 6r + 25 = 0 \Rightarrow$

$$r = \frac{-6 \pm \sqrt{36 - 25\cdot 4}}{2} = -3 \pm 4i.$$

$$\sqrt{36-100} = \sqrt{-64} = 8i.$$

$$y(x) = C_1 e^{-3x}\cos 4x + C_2 e^{-3x}\sin 4x$$

Higher order equations:

The method generalizes to

(*) $y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n = 0.$

$L = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$

Try solutions $\underbrace{y(x) = e^{rx}}$

$$\underline{r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n.}$$

(Defn) Auxiliary polynomial. $p(r)$

$p(r) = 0$ aux. equation.

$$p(r) = (r-r_1)^{m_1}(r-r_2)^{m_2} \cdots (r-r_k)^{m_k}$$

$m_1 + m_2 + \cdots + m_k = n.$, $r_1, \cdots r_k$ are complex numbers.

$$L = (D-r_1)^{m_1}(D-r_2)^{m_2} \cdots (D-r_k)^{m_k}.$$

$r_1 \cdots r_k$ complex numbers

- Case 1 r_1 real number.

$(D-r_1)^{m_1}$ has solutions

$$\underbrace{e^{r_1 x},\ xe^{r_1 x},\ x^2 e^{r_1 x}, \ldots, x^{m_1 - 1} \cdot e^{r_1 x}}_{m_1 \text{ linearly independent solutions}}$$

other real roots contribute similar solutions.

· case 2. if $r_1 = \alpha + \beta i$. $i^2 = -1$

then $\bar{r}_1 = \alpha - \beta i$ is also a solution.

$$\boxed{(D - r_1)^{m_1}(D - \bar{r}_1)^{m_1}}$$ has solutions

$$e^{\alpha x} \cdot \cos \beta x, \quad e^{\alpha x} \sin \beta x,$$

$$x e^{\alpha x} \cdot \cos \beta x, \quad x e^{\alpha x} \sin \beta x$$

$$\vdots$$

$$\underbrace{x^{m_1 - 1} e^{\alpha x} \cos \beta x, \quad x^{m_1 - 1} e^{\alpha x} \sin \beta x.}$$

$2m_1$ linearly independent solutions.

Collect all the solutions to

$$(D - r_i)^{m_i} \quad \text{or}$$

$$(D - r_i)^{m_i} (D - \bar{r}_i)^{m_i}.$$

we get a basis of $\{Ly = 0\}$

Ex: $y^{(4)}(x) - cy = 0, \quad c > 0.$

Aux. equation: $r^4 - c = 0.$

$$(r^2 + \sqrt{c})(r^2 - \sqrt{c})$$

$$= (r + c^{\frac{1}{4}})(r - c^{\frac{1}{4}})(r + c^{\frac{1}{4}} i)(r - c^{\frac{1}{4}} i)$$

$$r_1 = -c^{\frac{1}{4}}, \quad r_2 = c^{\frac{1}{4}}, \quad \underbrace{r_3 = -c^{\frac{1}{4}} i, \ r_4 = c^{\frac{1}{4}} i}_{\substack{\alpha + \beta i \\ \alpha - \beta i}} \ \text{complex conjugate.}$$

$$y(x) = C_1 e^{-c^{\frac{1}{4}} x} + C_2 e^{c^{\frac{1}{4}} x} + C_3 \cos(c^{\frac{1}{4}} x) + C_4 \sin(c^{\frac{1}{4}} x)$$

Initial conditions.

Ex: $y'' + 4y' + 4y = 0, \quad y(0) = 1, \ y'(0) = 4.$

Aux. poly: r^2+4r+4

$= (r+2)^2.$

$y(x) = C_1 e^{-2x} + C_2 x \cdot e^{-2x}$

$$\begin{cases} y(0) = C_1 + C_2 \cdot 0 = C_1 = 1 \\ y'(0) = C_1 \cdot (-2) e^{-2x}\Big|_{x=0} + C_2 \left(e^{-2x} + -2x e^{-2x}\right)\Big|_{x=0} \end{cases}$$

$= -2C_1 + C_2 = 4.$

$C_1 = 1. \quad C_2 = 6.$

$y(x) = e^{-2x} + 6x e^{-2x}$