Inhomogeneous equations + Annihilator method.

Recall: $P(D) = D^n + a_1 D^{n-1} + \cdots + a_n$

$a_1, \ldots a_n$ constant.

How to solve?

Aux. polynomial $P(r) = r^n + a_1 r^{n-1} + \cdots + a_n$.

Solve $P(r) = 0$. real root r. or

complex root $r = \alpha + \beta i$.

solutions are e^{rx}, $x e^{rx}$, $x^2 e^{rx}$, $\ldots$

$e^{\alpha x}\cos\beta x$, $x e^{\alpha x}\cos\beta x$, $\ldots$

$e^{\alpha x}\sin\beta x$, $x e^{\alpha x}\sin\beta x$, $\ldots$

Goal: Solve $P(D)y = \underline{F(x)}$ $(*)$

inhomogeneous.

Solutions are of the form

$$y(x) = \underbrace{y_p(x)}_{P(D)y_p = F.} + \underbrace{y_c(x)}_{P(D)y_c = 0.}$$

basis vectorspace.

Annihilator method.

Suppose we can find a polynomial operator

$A(D)$ s.t. $A(D)F = 0$.

Then $\underline{A(D)\,P(D)\,y} = A(D)\,F(x) = \underline{0}$. (**)

Solve the homogeneous equation.

(But not every solution of (**)

is a solution of (*).

$A(D)\,\square = 0$ does not mean

$\square = F(x)$)

Ex: Solve $(D+3)(D-3)y = 10e^{2x}$.

Step 1: Solve $(D+3)(D-3)y_c = 0$.

$y_c = C_1 e^{3x} + C_2 e^{-3x}$.

Step 2: Find. $\underline{A(D)\cdot(10e^{2x}) = 0}$

$A(D) = D-2$.

$\underline{A(D)\,P(D)}\cdot y = 0$

$(D-2)(D+3)(D-3)\,y = 0$.

$y = \underbrace{C_1e^{3x} + C_2e^{-3x}}_{y_c.} + \underline{A_0e^{2x}}$

General Theory tells us

$y = \underbrace{y_c}_{C_1e^{3x}+C_2e^{-3x}} + \underline{y_p}$

Step 3 Solve A_0.

(*) $(D+3)(D-3)(A_0e^{2x}) = 10e^{2x}$

$\underline{(D^2-9)(A_0e^{2x})} = 10e^{2x}$.

$$A_0\left(4\cdot e^{2x}-9\cdot e^{2x}\right)=10e^{2x}$$

$$-5A_0=10 \Rightarrow A_0=-2.$$

$$\boxed{y(x)=C_1e^{3x}+C_2e^{-3x}\underbrace{-2e^{2x}}_{y_p}.}$$

Ex: Find the general solution to

(*) $\quad (D-4)(D+1)\,y(x)=\underline{15e^{4x}}$

Step 1: $(D-4)(D+1)\,y_c(x)=0$

$$\Rightarrow y_c(x)=C_1e^{4x}+C_2e^{-x}$$

Step 2: an annihilator $A(D)$ of e^{4x} is

$$A(D)=D-4$$

$$(D-4)(D-4)(D+1)y(x)=0 \quad (**)$$

$$(D-4)^2(D+1)\,y(x)=0$$

$$y(x)=\underline{C_1e^{4x}}+\underline{C_2xe^{4x}}+\underline{C_3e^{-x}}$$

$$y_c(x)=C_1e^{4x}+C_2e^{-x}.$$

Step 3: $y_p = C_2xe^{4x}$

determine C_2,

$$y_p' = C_2\cdot(e^{4x}+4xe^{4x})$$

$$y_p''=C_2(\underbrace{4e^{4x}+4e^{4x}}_{8e^{4x}}+16xe^{4x})$$

$$(D-4)(D+1)=D^2-3D-4$$

$$(D-4)(D+1)\,y_p = C_2\Big(8e^{4x}+\underline{16xe^{4x}}$$
$$-3e^{4x}-\underline{3\cdot4xe^{4x}}$$

$$- \underline{4\ x}e^{4x})$$

$$= C_2 (5 e^{4x} - 0)$$

$$= 15 e^{4x}$$

$C_2 = 3$. $\qquad y_p = 3xe^{4x}$

$$y(x) = c_1 e^{4x} + c_2 e^{-x} + 3xe^{4x}$$

Summary: $\quad P(D)\, y(x) = e^{r_0 x}$.

Apply $(D - r_0) = A(D)$ on both sides

$p(r)$ may have a root r_0. with algebraic multiplicity m.

$$P(D) y_c = \cdots (D - r_0)^m \cdots = 0$$

has one more dim for solution space

$$A(D)\, P(D) y = \cdots (D - r_0)^{m+1} \cdots = 0$$

particular solution : $A_0 x^m \cdot e^{r_0 x} = y_p(x)$

Solve A_0 by plug $\underbrace{\ }_{y_p(x)}$ in $(*)$

Can also generalize to $F(x) = x^k \cdot e^{r_0 x}$

Ex: Find the general solution to

$(*)$ $(D^2 - 4D + 5)\, y(x) = 8x\, e^{2x} \cos x$

Step 1: Solve homogeneous equation.

$D^2 - 4D + 5$ has

Aux. poly $r^2 - 4r + 5 = 0$

$$r = \frac{4 \pm \sqrt{16 - 4 \cdot 5}}{2} = 2 \pm i.$$

$$y_c(x) = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x.$$

Step 2: Annihilator for $x\, \underline{e^{2x} \cos x}$

$e^{2x} \cos \beta x$. $\quad r_0 = 2 + \beta i$

$r_0 = 2+i.$ | $\bar{r}_0 = 2-i.$

$$\left((D-r_0)(D-\bar{r}_0)\right)^2$$

$$= \left[(D-(2+i))(D-(2-i))\right]^2 \leftarrow$$

$$= (D^2-4D+5)^2 = A(D)$$

$$A(D)\, x e^{2x} \cos x = 0$$

apply $A(D)$ to $P(D)y = f(x)$ $(*)$

$(**)$ $(D^2-4D+5)^3 y = 0$ order $=6$

$(D^2-4D+5)\, y_c = 0$ order $=2$.

4 more dims of solution space

$$y(x) = \underline{C_1 e^{2x}\cos x + C_2 e^{2x}\sin x} \longrightarrow y_c.$$

$$\left.\begin{array}{l} + A_0 x e^{2x}\cos x + B_0 x e^{2x}\sin x \\ + A_1 x^2 e^{2x}\cos x + B_1 x^2 e^{2x}\sin x. \end{array}\right\} \longrightarrow y_p.$$

Step 3. plug y_p into eq.

$$(D^2-4D+5)\cdot y_p = 8x\cos x.$$

↓ lots of simplifications.

$$(\underline{-2A_0+2B_1-4xA_1})\sin x + (\underline{2B_0+2A_1+4xB_1})\cos x$$

$$= 8x\cos x.$$

$$-2A_0+2B_1=0$$

$$-4A_1=0$$

$$2B_0+2A_1=0$$

$$4B_1=8.$$

$$\Rightarrow \quad A_1=0=B_0.$$

$$A_0=B_1=2$$

$$y_p = 2xe^{2x}\cos x + 2x^2e^{2x}\sin x.$$

$$y(x) = c_1 e^{2x}\cos x + c_2 e^{2x}\sin x +$$
$$2xe^{2x}\cos x + 2x^2 e^{2x}\sin x.$$

Annihilator method. $F(x) = x^k e^{r_0 x}$

$x^k e^{\alpha x}\cos\beta x$

$x^k e^{\alpha x}\sin\beta x.$

$\underline{A(D)}\, F(x) = 0$

order of $A(D)$ is m.

$A(D)\,P(D)\, y = 0$

compare

$P(D)\, y_c = 0$

get $\underline{m \text{ more solutions in the bases}}$ of solution spaces

Extra sol'n: $\underline{A_0 x e^{r_0 x} + A_1 x^2 e^{r_0 x} \cdots} = y_p$

solutions with m constan coefficients $A_0, A_1, \cdots$

Determine constants by plug in to (*).

Applications (to Spring-Mass system)

Chapter 8.5

spring constant (Hooke's law)

$$F = k \cdot y.$$

(damped) Spring-Mass equation: $y(t)$

damping constant

spring constant.

ODE $$y'' + \frac{c}{m} y' + \frac{k}{m} y = 0$$

from Newton's law

$k, c > 0$ are positive constants

$m > 0$

damped Spring mass equation with

external force. $F(t)$

(New equation) $$y'' + \frac{c}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t).$$

Notation $\omega_0 = \sqrt{\frac{k}{m}}$. ← frequency.

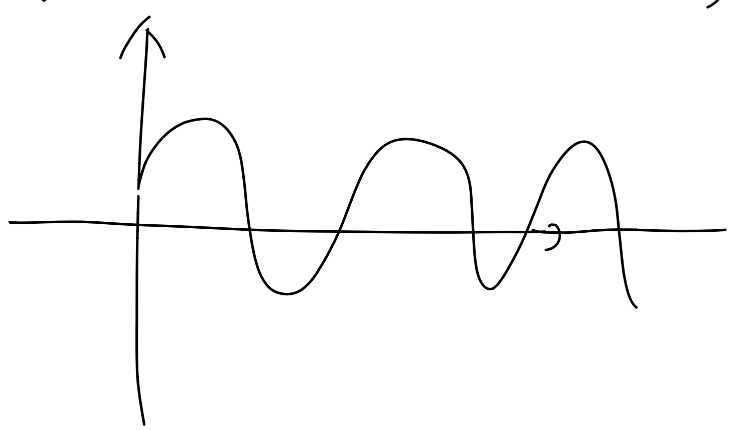
When $F = 0$.

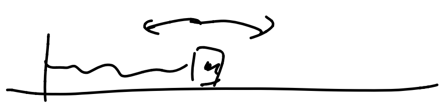
1. W/o damping $(c = 0)$

$$y'' + \frac{k}{m} y = 0 \qquad \underline{y'' + \omega_0^2 y = 0.}$$

Aux. poly $\quad r^2 + \omega_0^2 = 0, \quad r = \pm \omega_0.$

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$





2. Damping. $\quad c > 0.$

$$y'' + \frac{c}{m} y' + \frac{k}{m} y = 0$$

$$r^2 + \frac{c}{m} r + \frac{k}{m} = 0.$$

$$\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}$$

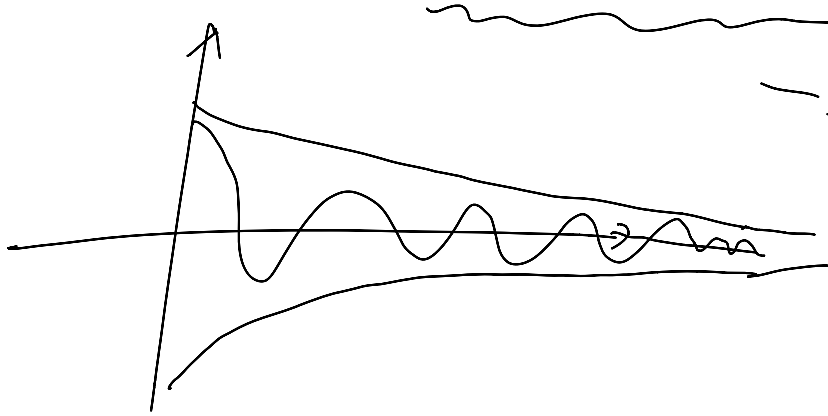
a) $\left(\frac{c}{m}\right)^2 - 4\frac{k}{m} < 0, \Rightarrow c^2 < 4km.$ (underdamping)

$$r = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

$$= -\left(\frac{c}{2m} \pm \frac{\sqrt{4km - c^2}}{2m} i\right) \qquad i^2 = -1.$$

$$\mu = \frac{\sqrt{4km - c^2}}{2m}.$$

$$y(t) = e^{-\frac{c}{2m}t}\left(c_1 \cos \mu t + c_2 \sin \mu t\right)$$

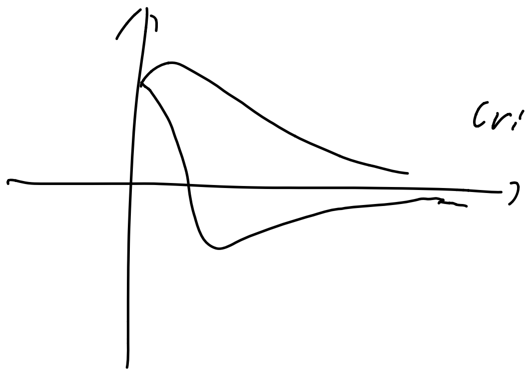


→ converge to zero.

b). $\left(\frac{c^2}{m}\right) - \frac{4k}{m} = 0$. $\to c^2 = 4km$. (critical damping

$r = -\frac{c}{2m}$

$y(t) = e^{-\frac{c}{2m}t}(c_1 + c_2 t)$.



critical damping / over damping

c) $c^2 > 4km$ (overdamping)

$$y(t) = e^{-\frac{c}{2m}t}\left(c_1 e^{\mu t} + c_2^{-\mu t}\right)$$

$$\mu = \frac{\sqrt{c^2 - 4km}}{2m}$$