Spring-Mass system. (Oscillations of a Mechanical system)

ODE: $y'' + \frac{c}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t)$ (*) — constants.

$y(t)$. $y(0) = y_0$, $y'(0) = v_0$.

Last time: external force $F(t) \equiv 0$.

Aux. poly: $r^2 + \frac{c}{m} r + \frac{k}{m} = 0$

Case 1: No damping. $c = 0$.

$$y(t) = c_1 \cos w_0 t + c_2 \sin w_0 t.$$

$$w_0 = \sqrt{\frac{k}{m}}.$$

$= A_0 \cos(w_0 t - \alpha)$

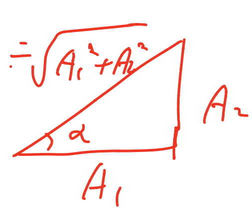
$A_0 = \sqrt{c_1^2 + c_2^2}$

Recall identities from trig.

$A_1 \cos\theta + A_2 \sin\theta$.

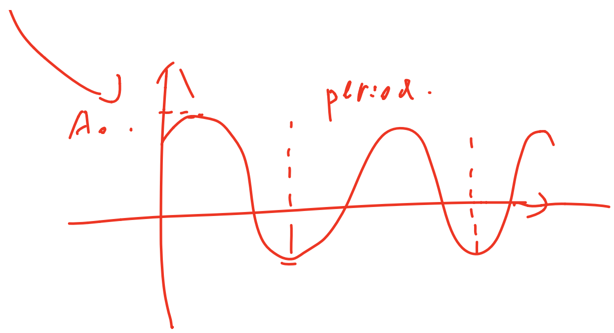
$= A_0 (\cos\alpha \cos\theta + \sin\alpha \sin\theta)$

$= A_0 \cdot \cos(\theta - \alpha)$.

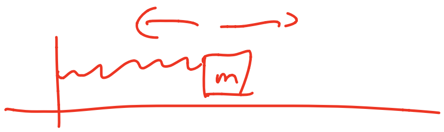


$\cos\alpha = \frac{A_1}{\sqrt{A_1^2 + A_2^2}} = \frac{A_1}{A_0}$

$\sin\alpha = \frac{A_2}{A_0}$



$$\text{period} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}.$$

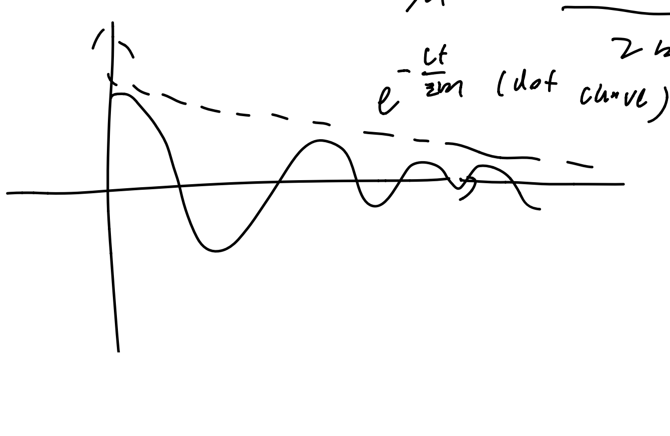


Case 2 Damping $c > 0$.

2.a) under damped $c^2 < 4km$. Aux. polynomial complex roots

$$y(t) = e^{-\frac{ct}{2m}}\left(c_1 \cos(\mu t) + c_2 \sin(\mu t)\right)$$

$$\mu = \frac{\sqrt{4km - c^2}}{2m}$$



b) critically damped $\quad c^2 = 4km.$

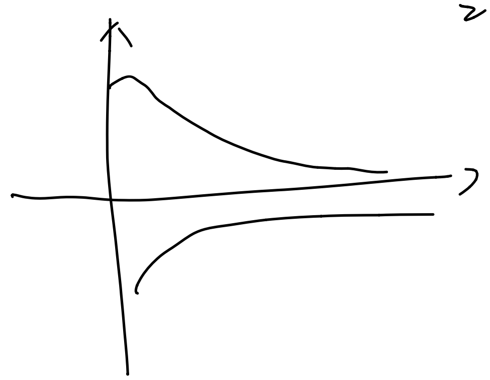
$$y(t) = e^{-\frac{ct}{2m}}(c_1 + c_2 t)$$

c) overdamped $\quad c^2 > 4km.$

$$y(t) = e^{-\frac{ct}{2m}}(c_1 e^{\mu t} + c_2 e^{-\mu t})$$

$$\mu = \frac{\sqrt{c^2 - 4km}}{2m}.$$

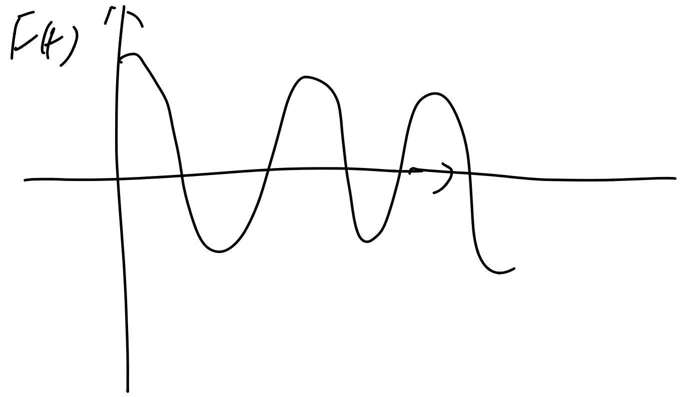
b). c)



$F(t) = 0$ homogeneous equation.

Today's class: we focus on

$F(t) = F_0 \cos \omega t$. F_0, ω are constants.



F(t) is periodic.

Case 1: No damping

homogeneous solutions:

$$y_c(t) = A_0 \cos(\omega_0 t - \phi).$$

$$y = y_c + y_p$$

Anni. of $F(t) = F_0 \cos \omega t$

$$A(D) = (D - \omega i)(D + \omega i)$$
$$= D^2 + \omega^2$$

$$P(D) = D^2 + \frac{c}{m} D + \frac{k}{m} = D^2 + \frac{k}{m}.$$

$A(D)\,P(D)$

If $\omega_0 \neq \omega$, i.e. $\frac{k}{m} \neq \omega^2$.

$$y_p = A_1 \cos \omega t + A_2 \sin \omega t.$$
$$= C_0 \cdot \cos(\omega t - \alpha)$$

$$y_p'' + \frac{k}{m} y_p = \frac{F_0}{m} \cos \omega t.$$

$$C_0 \left(-\omega^2 + \frac{k}{m} \right) \cos(\omega t - \alpha) = \frac{F_0}{m} \cos \omega t$$

$$C_0 = \frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad \alpha = 0$$

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

General solution $y(x) = A_0 \cdot \cos(\omega_0 t - \phi)$
$$+ \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

Look at case: $y(0) = y'(0) = 0$.

get: $y(0) = 0 \Rightarrow A_0 \cos\phi + \frac{F_0}{m(\omega_0^2 - \omega^2)} = 0$

$y'(0) = 0 \Rightarrow A_0 \omega_0 (-\sin(-\phi)) + \frac{F_0 \omega}{m(\omega_0^2-\omega^2)} \sin 0 = 0$

$\Downarrow$

$\sin\phi = 0 \Rightarrow \phi = 0$

$A_0 + \frac{F_0}{m(\omega_0^2 - \omega^2)} = 0 \qquad A_0 = \frac{-F_0}{m(\omega_0^2-\omega^2)}$

$$y(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}(\cos \omega t - \cos\omega_0 t)$$

when $\omega = \omega_0$. resonance

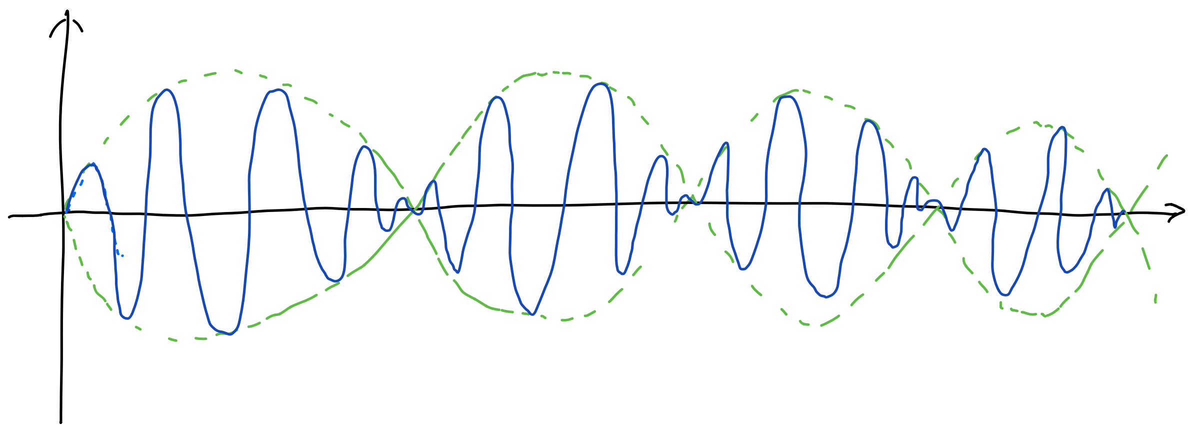
ω is related to $F(t)$

ω_0 is related to spring-mass system itself.

Recall trig identity: $\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$.

$A - B = \omega t$, $A + B = \omega_0 t$ $\Rightarrow \begin{cases} A = \frac{1}{2}(\omega_0 + \omega)t \\ B = \frac{1}{2}(\omega_0 - \omega)t. \end{cases}$

$$y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\Big(\underbrace{\frac{\omega_0 - \omega}{2}}_{\substack{\text{smaller} \\ \text{longer period.}}} t\Big) \sin\Big(\underbrace{\frac{\omega_0 + \omega}{2}}_{\substack{\text{not small} \\ \text{shorter period.}}} t\Big)$$



Resonance: $\omega = \omega_0$. $\quad y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t$.

$$A(D) = (D^2 + \omega_0^2)$$

$$P(D) = (D^2 + \omega_0^2)$$

$A(D)P(D)$ has repeated complex roots

$$\underbrace{i\omega_0}_{\text{algebraic multiplicity 2.}}, -i\omega_0$$

$$y_p = t(A_1 \cos w_0 t + A_2 \sin w_0 t)$$

$$(D^2 + w_0^2)\, y_p = \frac{F_0}{m} \cos w_0 t.$$

$$\Rightarrow A_1 = 0, \qquad A_2 = \frac{F_0}{2m w_0}.$$

General solution $y(t) = y_c + y_p$

$$= A_0 \cos(w_0 t - \phi)$$
$$+ \frac{F_0}{2m w_0} t \sin w_0 t.$$

$t \to +\infty.$

eventually, the spring will break.

Damping $c > 0$.

$$y'' + \frac{c}{m} y' + \frac{k}{m} = \frac{F_0}{m} \cos\omega t.$$

Aux poly $r^2 + \frac{c}{m} r + \frac{k}{m} = 0$ has not root $\pm \omega i$.

plug in.

$$y_p(t) = A_1 \cos\omega t + A_2 \sin\omega t.$$

long calculation $\Rightarrow$

$$y_p(t) = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2} \left[(k - m\omega^2) \cos\omega t + c\omega \sin\omega t \right]$$

general solution $y(x) = y_p + y_c$

$c > 0 \Rightarrow \lim_{t \to +\infty} y_c(t) \to 0$

"transient part"

steady state.

only see the part caused by external force.

New topic chapter 3.9.

Reduction of order method to find solutions to 2^{nd} order ODE given one (non-zero) solution to homo eqn.

Study $(*)$ $\quad y'' + a_1(x) y' + a_2(x) y = F(x).$

and suppose $y_1(x)$ solves the homogeneous equation $y'' + a_1(x) y' + a_2(x) y = 0.$

We know general solution

$$y(x) = \underbrace{y_c}_{c_1 y_1 + c_2 y_2} + y_p$$

We need y_2 and y_p

Ansatz: $y(x) = u(x) \cdot y_1(x)$. is a solution to $(*)$

Start to compute: $y' = u' y_1 + u y_1'.$

$$y'' = u'' y_1 + 2u' y_1' + u y_1''.$$

Substitute y into $(*)$

$$u'' y_1 + 2u' y_1' + u y_1'' + a_1(x)(u' y_1 + u y_1') + a_2(x)\, u \cdot y_1 = f(x).$$

$$u\underbrace{(y_1'' + a_1(x) y_1' + a_2(x) y_1)}_{=0} + u'' y_1 + 2u' y_1' + a_1(x) u' y_1 = f(x)$$

$$u'' y_1 + u'(2y_1' + a_1(x) y_1) = f(x).$$

$$(u')' y_1 + u'(2y_1' + a_1(x) y_1) = f(x)$$

Note: This is a 1^{st} order ODE for u'.

Call $w = u'$.

$$w' y_1 + w(2y_1' + a_1(x) y_1) = f(x).$$

$$W' + W\cdot\left(\frac{2y_1'}{y_1} + a_1(x)\right) = \frac{F(x)}{y_1}$$

Integration factor. $e^{\int^x \left(\frac{2y_1'(s)}{y_1(s)} + a_1(s)\right)ds}$. (mutiply on both sides)

$$\left(W\cdot e^{\int^x \left(\frac{2y_1'(s)}{y_1(s)} + a_1(s)\right)ds}\right)' =$$

$$\frac{F(x)}{y_1}\cdot e^{\int^x \left(\frac{2y_1'(s)}{y_1(s)} + a_1(s)\right)ds}$$

Notation $I(x) = e^{\int^x \frac{2y_1'(s)}{y_1(s)} + a_1(s)\,ds}$.

$$\frac{d}{dx}(W\cdot I) = \frac{F(x)}{y_1(x)}\cdot I(x)$$

$$W(x) = \frac{1}{I(x)}\left(\underline{\int^x \frac{I(s)F(s)}{y_1(s)}\,ds + C_1}\right).$$

$$U'(x) = W \Rightarrow$$

$$U(x) = \int^x \left(\frac{1}{I(t)}\int^t \frac{I(s)F(s)}{y_1(s)}\,ds\right)dt + C_1\int^x \frac{1}{I(s)}\,ds$$

$+ \underline{C_2}.$

general solution

$$y(x) = u(x)\cdot y_1(x)$$

$$= \underbrace{C_2 y_1(x) + C_1 \int^x \frac{1}{I(s)}\,ds}_{y_c(x)}$$

solution to homogeneous equation.

$$+ \underbrace{\int_0^x \left(\frac{1}{I(t)} \int^t \frac{I(s)F(s)}{y_1(s)}\,ds\right) dt}_{= y_p(x)}.$$

Use $F(x) = 0$ to find $y_c(x)$

Do Not memorize the proceding formula

Just use the idea $y(x) = u(x)\, y_1(x)$

and start computing.

Ex: Find the general solution to

$$xy'' - 2y' + (2-x)y = 0 \quad (x>0)$$

given one solution $y_1(x) = e^x$.

Look for $y(x) = u(x)\cdot e^x$

$$y' = u'\cdot e^x + u\,e^x$$

$$y'' = \underline{u''e^x + 2u'e^x + ue^x}$$

$$x\cdot(u''e^x + 2u'e^x + ue^x) - 2(u'e^x + ue^x) + (2-x)\cdot u\cdot e^x = 0$$

$$\Rightarrow \quad x\cdot u'' + 2u'(x-1) = 0.$$

$w = u'$, $\quad x\cdot w' + 2w(x-1) = 0.$

$$w' + \frac{2(x-1)}{x}\,w = 0.$$

Integration factor $I(x) = e^{2\int 1-\frac{1}{x}} = e^{2(x-\log x)}$

$$= e^{2x}\cdot e^{-2\log x}$$

$$= \frac{1}{x^2}\cdot e^{2x}$$

$$\left(w\cdot\frac{1}{x^2}e^{2x}\right)' = 0.$$

$$w = C_1\cdot x^2 e^{-2x} = u'.$$

$$u(x) = C_1 \cdot \int x^2 e^{-2x}\, dx + C_2$$

$$= -\frac{1}{4} C_1 \cdot (1+2x+2x^2) \cdot e^{-2x} + C_2$$

$y_1(x) = e^x$.

$$y(x) = C_1 e^{-x} (1+2x+2x^2) + C_2 e^x.$$