2^{nd} mini exam Chapter 8.

3^{rd} chapter 9.

4^{th} everything.

More examples of reduction of order.

Ex: $(D-5)^2 y(t) = 0$ $y(t) = C_1 e^{5t} + C_2 t e^{5t}$.

Aux. poly.

$(D-5)\underline{(D-5)y(t)} = 0$.

Find one solution by $(D-5)y(t) = 0$

$y_1(t) = e^{5t}$.

Ansatz: $y(t) = u(t)\cdot y_1(t) = u(t)\cdot e^{5t}$.

$y' = u'e^{5t} + 5ue^{5t}$

$y'' = u''e^{5t} + 5u'e^{5t} + 5u'e^{5t} + 25ue^{5t}$

$= \underline{u''e^{5t} + 10u'e^{5t} + 25ue^{5t}}$

$$(D-5)^2 y = 0$$

$$\Rightarrow \quad (D^2 - 10D + 25)y = 0.$$

$$u''e^{5t} + 10u'e^{5t} + 25ue^{5t} - 10(u'e^{5t} + 5ue^{5t})$$

$$+ 25ue^{5t} = 0.$$

$$u''e^{5t} + \underline{u'(10e^{5t} - 10e^{5t})}$$

$$+ u(\qquad\qquad\qquad) = 0$$

$$u'' = 0 \qquad \Rightarrow u = c_1 + c_2 t.$$

$$y(t) = u(t) \cdot y_1(t)$$

$$= e^{5t}(c_1 + c_2 t)$$

Ex. $\quad y'' - \tan x \, y' + 2y = 0. \quad (*)$

Guess $\quad y_1(x) = \sin x. \qquad \tan x = \dfrac{\sin x}{\cos x}$

$$y_1'' - \tan x \, y' + 2y = -\sin x - \tan x \cdot \cos x + 2\sin x = 0.$$

Apply reduction of order to (*).

$$y_2(x) = 1 - \sin x \log \frac{1+\sin x}{\cos x}$$

General solution:

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Recall: HW: Equidimensional equation

$$x^2 y'' + 3x y' + y = 0 \quad (*)$$

$$a_1 x^n y^{(n)} + a_2 x^{n-1} y^{(n-1)} + \cdots + a_{n+1} y = 0$$

$$y(x) = x^r,$$

$$x y' = x \cdot r x^{r-1} = r x^r.$$

$\vdots$

$$x^n y^{(n)} = \cdots = \square \cdot x^r.$$

constant related to r.

(*). Guess $y_1(x) = x^r$ solves (*)

$$y' = r x^{r-1}$$

$$y'' = r(r-1) x^{r-2}$$

$$r(r-1)x^r + 3r\,x^r + x^r = 0$$

$$\left(r(r-1) + 3r + 1\right) = 0$$

$$r^2 - r + 3r + 1 = r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0, \quad r = -1.$$

$y_1(x) = x^{-1}$ solves homogeneous eqn:

$$x^2y'' + 3xy' + y = 0.$$

Ex: Solve $x^2y'' + 3xy' + y = 4\ln x$. $(**)$ $x > 0$

$$y_1(x) = x^{-1}.$$

Try $\boxed{y(x) = u \cdot y_1(x)} = u \cdot x^{-1}$

$$y' = u'x^{-1} - ux^{-2}$$

$$y'' = u''x^{-1} - 2u'x^{-2} + 2ux^{-3}$$

$(**) \Rightarrow$

$$xu'' - 2u' + \cancel{2ux^{-1}}$$
$$+ 3u' - \cancel{3ux^{-1}} + \cancel{ux^{-1}} = 4\ln x.$$

(No term involving u)

$$xu'' + u' = 4\ln x.$$

$w = u'$. $\quad xw' + w = 4\ln x.$

$$\boxed{(xw)' = 4\ln x.}$$

Integration factor method. $\quad w' + \frac{1}{x}w = \frac{4\ln x}{x}.$

Multiply $\underline{e^{\int \frac{1}{x}} = e^{\ln x} = x}$. $\quad xw' + w = 4\ln x$

$$(xw)' = 4\ln x.$$

$$xw = 4x(\ln x - 1) + C_1.$$

$$u' = w = 4(\ln x - 1) + \frac{C_1}{x}.$$

$$u = 4(x(\ln x - 1) - x) + C_1 \ln x + C_2.$$

$$y(x) = u \cdot y_1$$

$$= \underbrace{4(\ln x - 2)}_{y_p} + \underbrace{C_1 \frac{\ln x}{x} + C_2 \frac{1}{x}}_{y_c}.$$

chapter 3.x.

complex numbers.

multiplication: $(1+2i)(3+5i)$

$$= 1\cdot 3 + 6i + 5i + 10\cdot(-1)$$

$$= -7 + 11i$$

inverse: $\dfrac{1+i}{1+2i} = \dfrac{(1+i)(1-2i)}{(1+2i)(1-2i)}$

$$= \frac{1+2+i-2i}{1^2+2^2 = 1-(2i)^2}$$

$$= \frac{1}{5}\cdot(3-i) = \frac{3}{5} - \frac{1}{5}i.$$

Euler formula: $e^{a+bi} = e^a\cdot(\cos b + (\sin b) i)$

$\underbrace{a}_{\text{Real part. } = a} + \underbrace{b}_{\text{Imaginary part. } = b} i.$

$\boxed{x^k e^{(a+ib)x}} \rightarrow$ $Re = x^k \cdot e^{ax}\cdot \cos bx$

$Im = x^k e^{ax}\cdot \sin bx.$

Thm: if $y(x) = u(x) + i v(x)$

$$y'' + a_1 y' + a_2 y = F(x) + i G(x)$$

then, $u'' + a_1 u' + a_2 u = F(x)$

$v'' + a_1 v' + a_2 v = G(x)$.

Pf: $y' = u'(x) + i v'(x)$

$y'' = u''(x) + i v''(x)$

Ex: $y'' + y' - 6y = 4 \cos 2x$. (*)

Method 1: $A(D) = \boxed{(D-2i)(D+2i)} = D^2 + 4$

$A(D)\ P(D)$

$P(D) = D^2 + D - 6 = (D+3)(D-2)$

$y_p = A_1 \cos 2x + A_2 \sin 2x$

(, (*) solve A_1, A_2.

Method 2: $\quad z'' + z' - 6z = 4e^{2ix} \quad (*)$

Real part solves

$$\boxed{y'' + y' - 6y = 4\cos 2x.}$$

$$A(D) = (D - 2i)$$

$$z_p = A_0 \cdot e^{2ix}$$

$$z_p' = 2i A_0 e^{2ix}$$

$$z_p'' = -4 A_0 e^{2ix}$$

$$A_0(-4 + 2i - 6)e^{2ix} = 4e^{2ix}$$

$$A_0 = \frac{4}{-4+2i-6} = \frac{2}{-5+i} = \frac{2\cdot(-5-i)}{5^2+1^2}$$

$$= \frac{-2(5+i)}{26} = -\frac{1}{13}(5+i)$$

$$y_p = \operatorname{Re} z_p = \operatorname{Re}\ -\frac{1}{13}(5+i)e^{2ix}$$

$$= \operatorname{Re}\ -\frac{1}{13}(5+i)(\cos 2x + i\sin 2x)$$

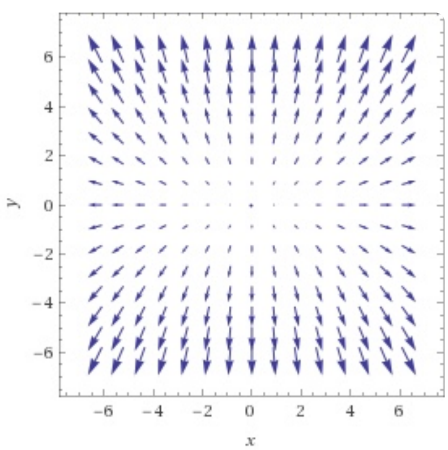
$$= -\frac{1}{13}(5\cos 2x - \sin 2x)$$

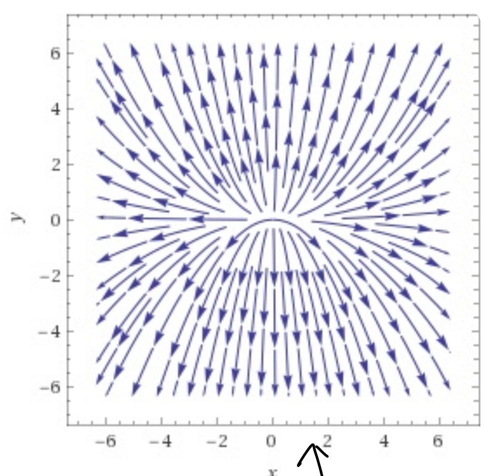
$$y = c_1 e^{-3x} + c_2 e^{2x} - \frac{1}{13}(5\cos 2x - \sin 2x)$$

Chapter 9 first order system of ODE's.

Ex: Vector field $\vec{F}(x_1, x_2) = \begin{pmatrix} 2x_1 \\ 4x_2 \end{pmatrix}$

Integral curve or flow line



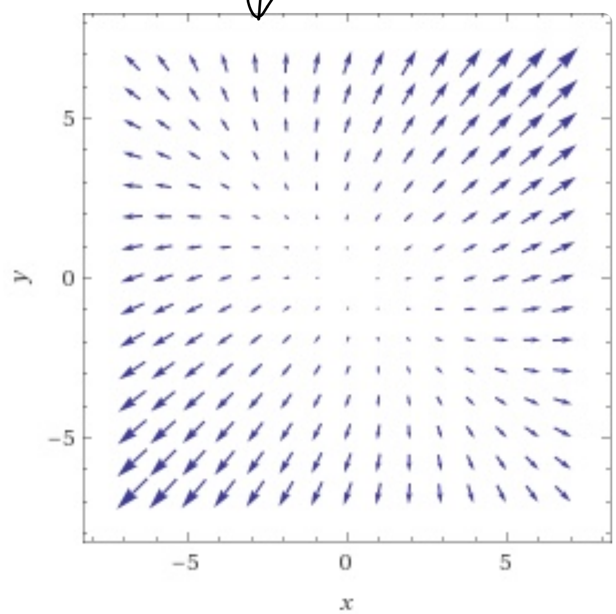


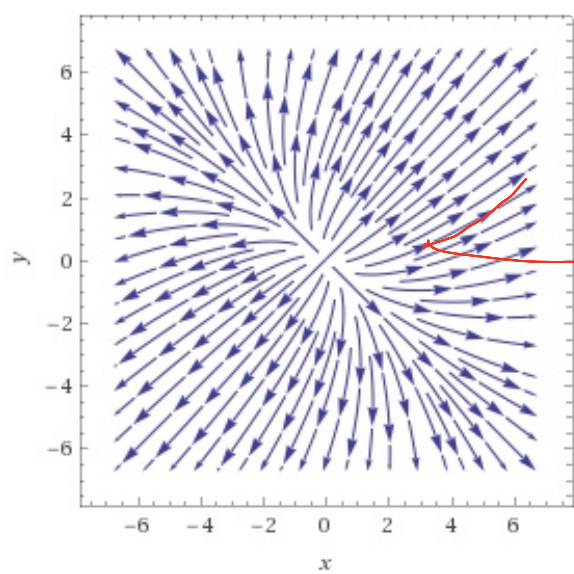
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}' = \begin{pmatrix} 2x_1 \\ 4x_2 \end{pmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1(t) = c_1 e^{2t}$$

$$x_2(t) = c_2 e^{4t}$$

$$\bar{F}'(x_1, x_2) = \begin{pmatrix} 3x_1 + x_2 \\ x_1 + 3x_2 \end{pmatrix}$$





Initial value $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}' = \begin{pmatrix} 3x_1 + x_2 \\ x_1 + 3x_2 \end{pmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1' = 3x_1 + x_2$$

$$x_2' = x_1 + 3x_2 .$$

$\uparrow$

1^{st} order system of ODEs

Defn: A 1^{st} order system of ODEs is of the form

$$(*)\begin{cases} x_1'(t) = a_{11}(t)\,x_1(t) + a_{12}(t)\,x_2(t) + \cdots + a_{1n}(t)\,x_n(t) + b_1(t) \\ x_2'(t) = a_{21}(t)\,x_1(t) + \cdots + a_{2n}(t)\,x_n(t) + b_2(t) \\ \vdots \\ x_n'(t) = a_{n1}(t)\cdot x_1(t) + \cdots + a_{nn}(t)\,x_n(t) + b_n(t) \end{cases}$$

(*) is homogeneous if

$$b_1(t) = b_2(t) = \cdots = b_n(t) = 0$$

Convenient to write in vector form

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & \cdots & \vdots \\ \vdots & \ddots & a_{nn}(t) \end{bmatrix}$$

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad b(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

$$(*) \qquad x'(t) = A(t) \cdot x(t) + b(t)$$

Rmk: higher order ODE is also related to 1^{st} order system of ODEs.

$$y'' + y' - 6y = e^x \qquad y(x)$$

$$y_1(x) = y(x)$$

$$y_2(x) = y'(x)$$

$$\begin{cases} y_1' = y_2 . \\ y_2' = y'' = 6y - y' + e^x = 6y_1 - y_2 + e^x \end{cases}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}}_{A(x)} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ e^x \end{pmatrix}}_{b(x)}$$

n^{th}-order ODE $\Rightarrow$ 1^{st}-order ODE system with n equations

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Notation: $V_n([a,b])$ is the vector space of all n-vector value funtions defined on $[a,b]$.

$$V_n([a,b]) = \left\{ \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \middle| \ x_1, \dots x_n \text{ are functions defined on } [a,b] \right\}$$

$A(t)$ defines a linear transformation.

$$T: V_n([a,b]) \longrightarrow V_n([a,b])$$

$$T\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = A(t)\cdot x(t)$$

Another linear transformation on

$V_n^\infty([a,b])$ space of smooth n-vector valued functions

$$T: V_n^\infty([a,b]) \longrightarrow V_n^\infty([a,b])$$

$$T(x(t)) = \underline{x'(t) - A(t)\cdot x(t)}$$

Solutions to homogeneous ODE system.

$x'(t) = A(t)\cdot x(t)$ is

$\ker T$. hence a subspace.

Solving homogeneous system.

is related to find dim ker T

and basis of ker T.

Thm: dim ker $T = n$.

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linear independence / dependence of elements in

$V_n([a,b])$.

Def: Let $\mathbb{X}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}$

$\mathbb{X}_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix} \cdots \mathbb{X}_n(t)$

The Wronskian $W[\mathbb{X}_1(t), \cdots \mathbb{X}_n(t)]$

is $\det \begin{bmatrix} | & | & & | \\ \mathbb{X}_1(t) & \mathbb{X}_2(t) & \cdots & \mathbb{X}_n(t) \\ | & | & & | \end{bmatrix}$

Whether $\{\mathbb{X}_1, \cdots \mathbb{X}_n\}$ is linearly independent

is related to solve $C_1 \mathbb{X}_1(t) + \cdots + C_n \mathbb{X}_n(t) = 0$

$C_1 \cdots C_n$ are constants.

Fact: If $W[\mathbb{X}_1(t), \cdots \mathbb{X}_n(t)](t_0) \neq 0$ at

some point $t_0 \in [a, b]$,

then $\{\mathbb{X}_1(t), \cdots \mathbb{X}_n(t)\}$ is linearly independent.

Ex: $\mathbb{X}_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$, $\mathbb{X}_2(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$$W[\mathbb{X}_1, \mathbb{X}_2] = \begin{vmatrix} \sin t & \cos t \\ \cos t & \sin t \end{vmatrix}$$

$$= \sin^2 t - \cos^2 t = -\cos 2t$$

is not always 0.

$\{\mathbb{X}_1, \mathbb{X}_2\}$ linearly independent.

Thm tells us we need to find n solutions

$\mathbb{X}_1, \dots \mathbb{X}_n$ and check $W[\mathbb{X}_1, \dots, \mathbb{X}_n) \neq 0$ at some point t_0.

All the solutions to homogeneous system of ODEs are of the form

$$C_1\mathbb{X}_1 + C_2\mathbb{X}_2 + \dots + C_n\mathbb{X}_n.$$

Black box thm:

1^{st} order system of ODEs has a unique solution with fixed initial condition

$$x(t_0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Ex:
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$\mathbb{X}(t) = A \cdot \mathbb{X}(t). \quad (*)$$

key idea: If v, λ are e-vector, e-value pair.

then $x(t) = e^{\lambda t} \cdot v$ solves $(*)$

check $x'(t) = \lambda e^{\lambda t} v$

$$= e^{\lambda t} \cdot (\lambda v)$$

$$= e^{\lambda t} \cdot A \cdot v.$$

$$= A(e^{\lambda t} v)$$

$$= A \cdot x(t).$$

If we have a basis of e-vectors $\{v_1, v_2\}$.

then $\{e^{\lambda_1 t} v_1, e^{\lambda_2 t} v_2\}$ spans the solution space

because $W[e^{\lambda_1 t} v_1, e^{\lambda_2 t} v_2]$

$$= \begin{vmatrix} | & | \\ e^{\lambda_1 t} v_1, & e^{\lambda_2 t} v_2 \\ | & | \end{vmatrix} = e^{(\lambda_1 + \lambda_2)t} \begin{vmatrix} | & | \\ v_1, & v_2 \\ | & | \end{vmatrix}$$

$$\neq 0.$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_1 = 4, \qquad \lambda_2 = 2$$

$$x(t) = c_1 e^{4t} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Conclusion: A constant, diagonalizable matrix. $b(t) = 0$.

Solutions are

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \ldots + c_n e^{\lambda_n t} v_n.$$

$\{v_1, v_2 \ldots v_n\}$ basis consisting of e-vectors for A.