Lec 2 1/21/20

Today : system of equations
row reduction
Gaussian elimination,
reduced row-echelon form.

Ex:
$$\left[\begin{array}{ccccc|c} 1 & 1 & 3 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 2 & -4 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 & -6 & 9 \end{array}\right]$$

$$\xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & 1 & 0 & -2 & 3 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(non-pivot columns) free variable columns

Back with variables

$$x_1 + x_3 - 2x_5 = 3$$
$$x_2 + 2x_3 + x_5 = -2$$
$$x_4 - x_5 = 1$$

Introduce extra equations
(corresponding to free vars x_3, x_5)

$$x_3 = x_3, \qquad x_5 = x_5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$x_3, x_5 \in \mathbb{R}$$

Reduced Row Echelon Form (r.r.e.f)

The "optimal" row reduced form

— can "read off" solutions

What it is:

(a) 1^{st} non-zero entry in each row is 1.

(b) entries above, below a pivot in same column are 0

(c). If some row has a pivot, another row above has a pivot further to the left.

The r.r.e.f of a matrix is <u>unique</u>, i.e. different sequences of row reduction lead to the same solutions.

Ex: Some matrices in r.r.e.f.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Matrices <u>not</u> in r.r.e.f.

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Ex: (reading off soln's from r.r.e.f)

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & 7 \end{bmatrix}$$

free variables.

$$x_1 + 3x_4 = 1.$$

$$x_2 = x_2$$

$$x_3 + 3x_4 = 7$$

$$x_4 = x_4.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 7 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$s, t \in \mathbb{R}$,

rename $x_2 = s$

$x_4 = t$.

Notice: Each column is either a pivot column or a free variable column, i.e.

$$\boxed{\text{\# of pivots} + \text{\# free variables} = \text{\# of columns.}}$$

This eqn is known as the Rank-Nullity theorem.

(very important).

Def: The rank of a matrix A is the number of pivots in r.r.e.f (A).

Facts: • An $m \times n$ matrix has rank $\le \min(m, n)$

• memorize the rank-nullity theorem.

Matrix multiplication:

$$\underset{m \times q}{A} \times \underset{q \times n}{B} = \underset{m \times n.}{AB}$$

must match.

Defn: (matrix multiplication).

Given A $m \times q$, B $q \times n$, the (i,j)-th entry of AB is

$$r_i \cdot c_j = \sum_{l=1}^{q} A^i_l B^l_j$$

row entry

column entry.

$$\begin{bmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — r_m — \end{bmatrix} \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} r_i \cdot c_j \end{bmatrix}.$$

In index notation:

$$(AB)^i_j = \sum_{l=1}^{q} A^i_l B^l_j .$$

Associativity: $(AB)C = A(BC)$

In general, not commutative.

- AB allowed.
  BA may not be allowed

- AB, BA different size.
  A 3×2 $\quad$ AB 3×3
  B 2×3. $\quad$ BA 2×2.

- A, B $m \times m$,

$AB \neq BA$ in general.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Scalar multiple of a matrix

$$(a \cdot A)^i_j = a \cdot A^i_j$$

Addition A $n \times m$.

B $n \times m$.

$$(A+B)^i_j = A^i_j + B^i_j$$

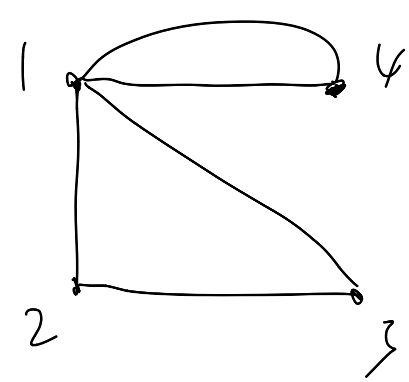
Some formulas

$$a(A+B) = (aA + aB).$$

$$(A+B)\cdot C = AC + BC$$

$$C\cdot(A+B) = C\cdot A + C\cdot B$$

Application to network theory:



Let G be a graph. (No direction for each edge).

Defn: The Ajacency matrix A for a graph with n vertices has ij entry A_{ij} = # of edges connecting vertex i to vertex j.

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

(A is symmetric, $A = A^T$)

Q: How many length k paths are there from vertex i to vertex j.

Ans: The answer is ij entry of

$$\underbrace{A^k}_{\text{k-th power}} = \underbrace{A \cdot \cdots \cdot A}_{k}$$

length 2 paths from i to j

$$= \underbrace{(\text{\# of edges from } i \text{ to } l) \cdot (\text{\# of edges from } l \text{ to } j)}_{\text{sum over all intermediate vertices } l.}$$

$$= \sum_{l=1}^{n} A^i_l \cdot A^l_j = (A^2)^i_j .$$

Defn: The transpose A^T is a $m \times n$ matrix with

$$(A^T)^i_j = A^j_i$$

$$\underset{A}{\begin{bmatrix} - & r_1 & - \\ - & r_2 & - \end{bmatrix}} \qquad \underset{A^T}{\begin{bmatrix} | & | \\ r_1 & r_2 \\ | & | \end{bmatrix}}$$

$$\begin{bmatrix} 0 & -2 & 6 \\ 1 & 3 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 6 & 0 \end{bmatrix}$$

Fact: If $A \cdot B$ is defined,

then $(AB)^T = B^T \cdot A^T$.