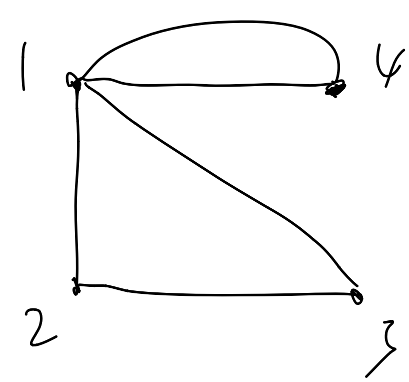
Some formulas

$$a(A+B) = (aA + aB).$$

$$(A+B)\cdot C = AC + BC$$

$$C\cdot(A+B) = C\cdot A + C\cdot B$$

Application to network theory:



Let G be a graph.
(No direction for each edge).

Defn: The Ajacency matrix A for a graph with n vertices

$$\underbrace{A^k}_{\text{k-th power}} = \underbrace{A \cdot \cdots \cdot A}_{k}$$

length 2 paths from i to j

$= (\# \text{ of edges from } i \text{ to } \ell)$

$\cdot (\# \text{ of edges from } \ell \text{ to } j)$

sum over all intermediate vertices ℓ.

$$= \sum_{l=1}^{n} A^i_l \cdot A^l_j = (A^2)^i_j .$$

Defn: The transpose A^T is a $m \times n$ matrix with

$$(A^T)^i_j = A^j_i$$

$$\begin{bmatrix} - & r_1 & - \\ - & r_2 & - \end{bmatrix} \qquad \begin{bmatrix} | & | \\ r_1 & r_2 \\ | & | \end{bmatrix}$$

A $\qquad$ A^T

$$\begin{bmatrix} 0 & -2 & 6 \\ 1 & 3 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 6 & 0 \end{bmatrix}$$

Fact: If $A \cdot B$ is defined,

then $(AB)^T = B^T \cdot A^T$.

Useful facts:

Identity matrix, $I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

$$I \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$I \cdot A = A.$$

$$A \cdot \begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} Ac_1, & Ac_2, & Ac_3 \end{bmatrix}$$

Matrix inverse. (2.6).

Defn: Let A be an $n \times n$ matrix. If there is an matrix A^{-1} satisfying

$$AA^{-1} = A^{-1}A = I_n.$$

we say A^{-1} is the inverse of A and say A is invertible

Facts

- Notion of inverse only defined for square matrices
- If A has an inverse it is unique.

pf: If B and C are inverse of A.

$$BA = AB = I$$
$$CA = AC = I.$$

$$B = BI = BAC = (BA)C = I \cdot C$$
$$= C.$$

Ex:

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix},$$

$$A^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}.$$

check $A \cdot A^{-1} = A^{-1} \cdot A = I_2$.

Two theorems:

① Solve $Ax=b$ by A^{-1}

② $\operatorname{rk} A = n \iff A$ invertible.

③ Compute A^{-1} by row reduction.

①: $Ax=b$ multiply both sides by A^{-1}. $(A^{-1}A)x = A^{-1}b$

$$x = A^{-1}b.$$

$$[A \mid b] \xrightarrow[\text{reduction}]{\text{row}} [I_n \mid A^{-1}b]$$

(2) "$\Rightarrow$" A invertible $\Rightarrow$

$Ax=b$ has unique solution

$\Rightarrow$ rref of A has no free variables.

"$\Leftarrow$" $\operatorname{rank}(A)=n$ $\Rightarrow$

want to find B

$$B=\begin{bmatrix} | & | & | & & | \\ c_1 & c_2 & c_3 & \cdots & c_n \\ | & | & | & & | \end{bmatrix}$$

such that

$AB=I_n$, $Ac_1=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$

$Ac_2=\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}$ $\cdots$ $Ac_n=\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$

$$AC_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ has a unique solution

$$AC_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ has a unique solution.

$\vdots$

$$AC_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$ has a unique solution.

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \xrightarrow{\text{row reduction}}$$

$$\left[\begin{array}{c|c} I_n & * \end{array}\right]$$ $*$ is the solution to $AC_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

So we can find B. s.t.

$AB = I_n$.

$ABA = (AB)A = A = A \cdot I_n$

$A(BA - I_n) = 0$. $\quad BA - I_n = \begin{pmatrix} | & | & \\ d_1 & d_2 & \cdots \\ | & | & \end{pmatrix}$

since $Ad_i = 0 \Rightarrow d_i = 0$.

so $BA - I_n = 0 \Rightarrow BA = I_n$.

3 How to compute B.

$$[A \mid I_n] \xrightarrow{\text{row reduction}} [I_n \mid A^{-1}].$$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccccc} 1 & & & 0 & 1 & -1 \\ & 1 & & 2 & -2 & -1 \\ & & 1 & -1 & 1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

some formulas:

① $(AB)^{-1} = B^{-1}A^{-1}$

Pf: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A \cdot I_n A^{-1} = AA^{-1} = I_n$

$(B^{-1}A^{-1})\cdot(AB) = B^{-1}(A^{-1}A)B$

$= I_n.$

② $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$