Remarks on matrix multiplication:

$$A = \begin{pmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

$$AB = \left(A \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \; A \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right)$$

$$= \left(c_1 + 2c_2 + 3c_3 \, , \; 4c_1 + 5c_2 + 6c_3 \right)$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} (1\,2\,3) \cdot B \\ (4\,5\,6) \cdot B \end{pmatrix}$$

$$= \begin{pmatrix} r_1 + 2r_2 + r_3 \\ 4r_1 + 5r_2 + 6r_3 \end{pmatrix}$$

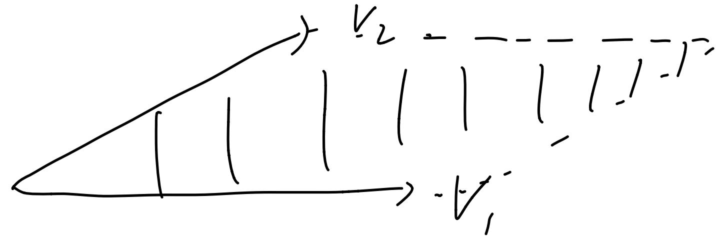
$A \cdot B$ each column is a linear combination of columns of A, with coefficients given by B.

each row is a linear combination of rows of B, with coefficient given by A.

Determinants. square matrix.

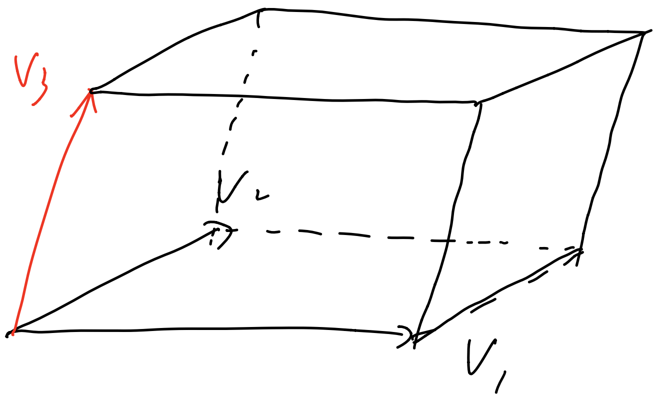
Defn: If A is an $n \times n$ matrix. $\det(A)$ is defined to be the (oriented) volume of the parallelepiped formed by the row vectors of A.

$$\det A = |A| = \det\begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix} = \det(v_1, \dots, v_n).$$



in $\mathbb{R}^2$, $\pm|v_1 \times v_2|$

In $\mathbb{R}^3$.



$(v_1 \times v_2) \cdot v_3$

$v_1 \cdot (v_2 \times v_3)$

Why determinants useful?

Thm: A is invertible $\iff$ $\det A \neq 0$.

In $\mathbb{R}^2$, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= ad - bc.$

In $\mathbb{R}^3$. $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot \begin{pmatrix} e & f \\ h & i \end{pmatrix}$

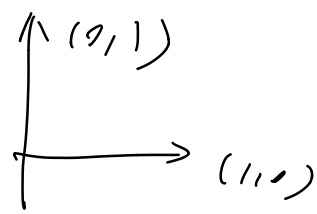
$- b\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \begin{pmatrix} d & e \\ g & h \end{pmatrix}$

$$= aei + dhc + gbf$$
$$- ceg - fha - idb.$$

How to compute in general.

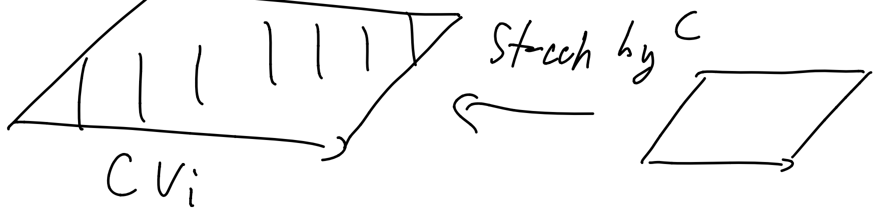
Fundamental properties:

(1) $\det(I_n) = 1$

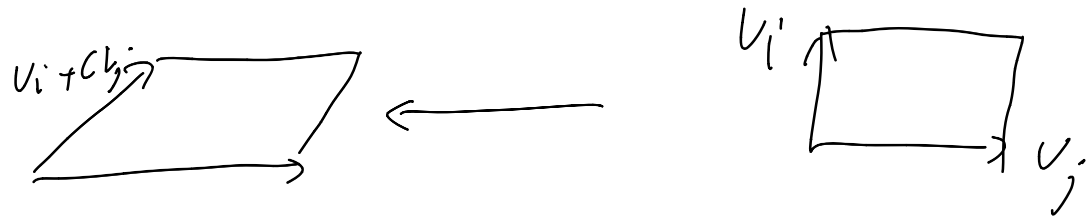


(2) $\det(v_1, \dots, cv_i, \dots, v_n) =$

$$c \cdot \det(v_1 \cdots v_i \cdots v_n)$$



(3)



$$\det(v_1, \cdots, v_i + c v_j, \cdots, v_n)$$

$$= \det(v_1, \cdots, v_i, \cdots, v_n)$$

To compute $\det(A) = \det\begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}$

now reduce A to echelon form, keeping track of property (1)

Ex:

$$\begin{vmatrix} 1 & 3 & 4 \\ 1 & 0 & 2 \\ 3 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ 0 & -3 & -2 \\ 3 & 3 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 4 \\ 0 & -3 & -2 \\ 0 & -6 & -9 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ 0 & -3 & -2 \\ 0 & 0 & -5 \end{vmatrix}$$

$$= 1 \cdot (-3) \cdot (-5) = 15$$

(4) Switching two rows multiplies $\det A$ by (-1)

$$\det(v_1 \cdots v_i, \cdots v_j, \cdots v_n)$$

$$= -\det(v_1 \cdots v_j, \cdots v_i, \cdots v_n)$$

pf: $\det(v_1 \cdots v_i + v_j, \cdots, v_j, \cdots, v_n)$

add $-(v_i+v_j)$ to v_j.

$= \det(v_1 \cdots v_i+v_j, \cdots -v_i, \cdots v_n)$

add $(-v_i)$ to v_i+v_j.

$= \det(v_1 \cdots v_j, \cdots -v_i, \cdots v_n)$

$= -\det(v_1 \cdots v_j, \cdots v_i, \cdots v_n)$.

Ex:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

subtract v_1 from all others.

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} = (-1)^4 = 1.$$

Ex:

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 5 \\ 3 & 4 & 5 & 5 & 5 \\ 4 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{vmatrix} \quad \text{Substract each row from the previous} \quad \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Switching r_1, r_5 and r_2, r_4

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{vmatrix} = 5.$$

Cofactor Expansion.

Method: a) Pick a row or a column.

$$\left| \begin{matrix} + & - & + & \cdots & - & \\ - & + & - & + & \cdots & - \\ + & - & + & - & \cdots & - \end{matrix} \right.$$

b) Multiply entries with the corresponding signs by the $(n-1)\times(n-1)$ determinant formed by deleting the row and column of the present entry.

Ex:

$$\begin{array}{cccc} + & - & + & \\ \end{array}$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 11 & 0 & 2 \\ 0 & -1 & 0 & 6 \\ 0 & 1 & 0 & 17 \end{vmatrix}$$

$$= 3 \cdot \begin{vmatrix} 5 & 11 & 2 \\ 0 & -1 & 6 \\ 0 & 1 & 17 \end{vmatrix}$$

$$= 3 \cdot 5 \cdot \begin{vmatrix} -1 & 6 \\ 1 & 17 \end{vmatrix} = 3 \cdot 5 \cdot (-23)$$

More facts

① $|AB| = |A| \cdot |B|$.

② $|A^{-1}| = \dfrac{1}{|A|}$

Pf: $|A^{-1} \cdot A| = |A^{-1}| \cdot |A|$

$= |I_n| = 1$

③ $|A| = |A^T|$.