$|A| = 0 \iff A$ is not invertible

$\iff \text{rref}(A) \neq I_n$

$\iff \text{rank}(A) < n$.

$\iff$ rref(A) has at least one free variable

$\iff Ax = 0$ has infinitly many solutions (non-zero solutions)

Ex: Find the equation of the plane passing through $(1,0,0)$, $(0,1,0)$, $(0,3,1)$

Equation of the plane is

$$\begin{cases} c_1 x + c_2 y + c_3 z + c_4 = 0. \\ c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 + c_4 = 0 \\ c_1 \cdot 0 + c_2 \cdot 1 + c_3 \cdot 0 + c_4 = 0 \\ c_1 \cdot 0 + c_2 \cdot 3 + c_3 \cdot 1 + c_4 = 0 \end{cases}$$

$$\iff \begin{bmatrix} x & y & z & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has at least one non-zero solution

(C_1, C_2, C_3, C_4) (zero already a solution, so infinitely many solution)

so

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix} - y \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$+ z \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 0.$$

$$x + y - 2z - 1 = 0$$

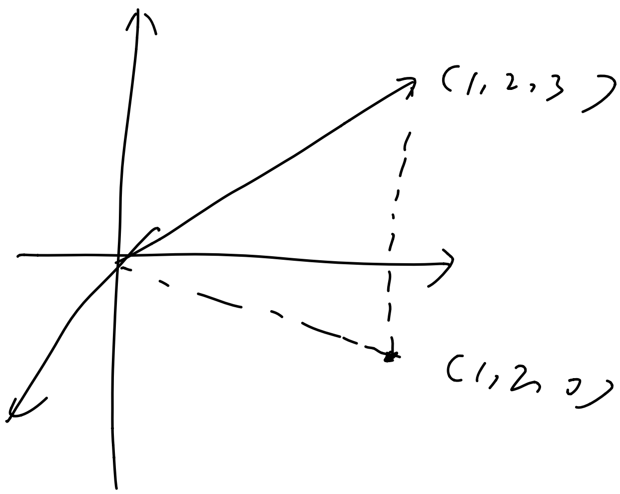
Vector spaces.

(1) $\mathbb{R}^n$

(2) the solutions to a homogeneous ordinary equation

Why general vector spaces:

rephrase a question like "what degree-3 polynomial best approximates $\sin x$ on $[0, \pi]$?" into a question about vector spaces analogous to "what vector in $\{z = 0\}$ is closest to the vector $(1, 2, 3)$?" which is easier to answer.



Def: A nonempty set V with a $\boxed{\text{rule}}$ for scalar multiplication and addition is a <u>vector space</u> if it satisfies the following:

① V is closed under addition.
i.e. given $v, w \in V$, $v+w \in V$

② V is closed under scalar multiplication.
i.e. given any scalar c. $cv \in V$.

Rmk: "rule for addition / scalar mult." on p.253. 10 properties.

Ex: ① $\mathbb{R}^n$

② $V = \{\text{all functions } f: \mathbb{R} \to \mathbb{R}\}$.

$:= \mathcal{F}(\mathbb{R}, \mathbb{R})$

$$(f+g)(x) = f(x) + g(x)$$

$$(cf)(x) = c \cdot f(x)$$

③ $M_{m\times n} = \{\text{All } m\times n \text{ matrices with entries in } \mathbb{R}\}$.

is a vector space with componentwise addition and scalar multiplication

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1+a, & 2+b, & 3+c \\ 4+d, & 5+e, & 6+f \end{bmatrix}$

$c \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} c, & 2c, & 3c \\ 4c, & 5c, & 6c \end{bmatrix}$

(4) $V = \{$ all functions $f : \mathbb{R} \to \mathbb{R}$ solving $f'' + \frac{k}{m} f = 0 \}$.

is a vector space

why?

$f \in V$, $g \in V$, $f'' + \frac{k}{m} f = 0$

$g'' + \frac{k}{m} g = 0$.

$(f+g)'' + \frac{k}{m}(f+g) = (f'' + \frac{k}{m} f) + (g'' + \frac{k}{m} g) = 0$

so $f+g \in V$.

$(c \cdot f)'' + \frac{k}{m}(cf) = c \cdot (f'' + \frac{k}{m} f) = 0$.

so $c \cdot f \in V$

(5) $V = \{$all polynomials of degree $\leq k\}$

$$V = \{ f = a_0 + a_1 x + \cdots + a_k x^k \}.$$

$$\cong \mathbb{R}^{k+1}$$

Non example:

$$V = \{ f : \mathbb{R} \to \mathbb{R} \text{ solving } f'' + \frac{k}{m} f^2 = 0 \}.$$

$f \in V$, $g \in V$.

$$(f+g)'' + \frac{k}{m}(f+g)^2$$

$$= \left(f'' + \frac{k}{m} f^2\right) + \left(g'' + \frac{k}{m} g^2\right) + 2\frac{k}{m} f \cdot g$$

$$= 2\frac{k}{m} f \cdot g$$

$f \cdot g$ is not zero in general.

Subspaces

Defn: A subset $W \subset V$ is a subspace if W

a) W is closed under addition
$\forall w_1, w_2 \in W, \quad w_1 + w_2 \in W$

b) W is closed under scalar multiplication
$\forall c \in \mathbb{R}, \ w \in W, \quad c \cdot w \in W$

Example of subspaces:

① The plane $2x + 3y - z = 0$

② $S = \{ f : \mathbb{R} \to \mathbb{R}$ satisfying $f(1) = 0 \}$.

W itself is a vector space.

Important fact: A vector space must have "0". $0+v=v+0=v$

Ex: The set $\{(x,y) \mid 2x-y=3\}$ is not a subspace of $\mathbb{R}^2$ because $(0,0) \notin W$.

Q: Are the following subspaces?

(1) Row reduced echlon matrices (No)

(2) Polynomials of degree exactly 4 (No)
"0" is not in it.

(3) Solutions to $f''+3f=1$. (No).
"0" is not in it

(4) Row echlon matrices (No).

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

(5) All upper triangular 4×4 matrices.

(Yes).