Subspaces: ① closed under addition
② closed under scalar multiplication.
Fact: $0 \in W$.

Warm up:

① {All reduced row echelon form} = W

No. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W$. $2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \notin W$

② {All row echelon form}

No. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \notin W$.

③ {All solutions to $f'' + 3f = 1$.} = W

No: $0 \notin W$.

(4) {All upper triangular matrices}.

$$\begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$

Closed under addition.

Constraints are something $= 0$

(5) {All solutions to $f'' + f = 0$}.

Yes.

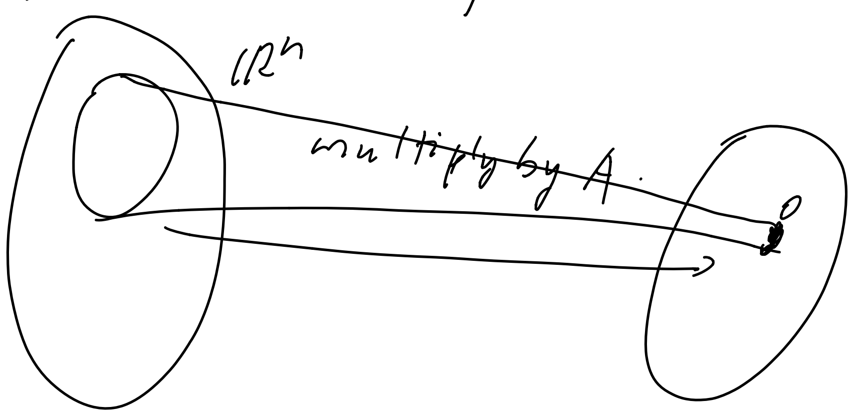
Homogeneous linear systems give subspaces

$Ax = 0$

Defn: The kernel, denoted by $\ker(A)$ of an $m \times n$ matrix A is

$\underline{\ker A} := \{ v \in \mathbb{R}^n \mid Av = 0 \}$

also called null space



Fact: $\ker A$ is a subspace of $\mathbb{R}^n$.

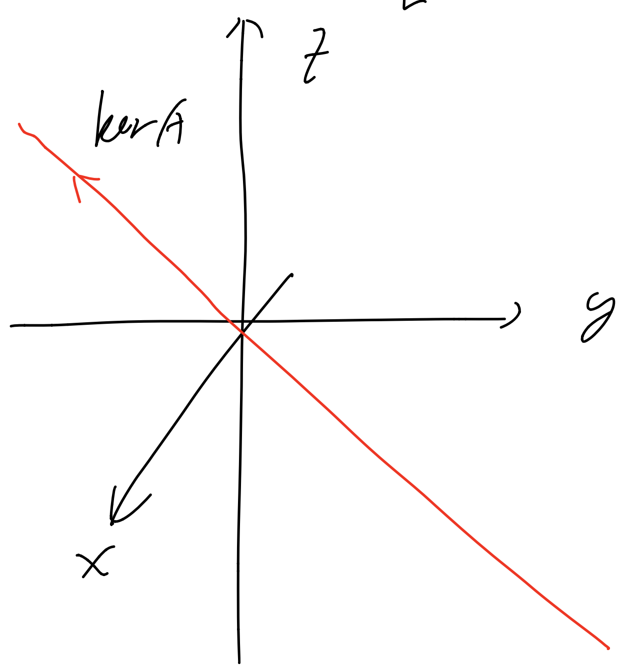
Ex: What is $\ker \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$?

Looking for solutions of $Av = 0$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solutions are

$$v = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad t \in \mathbb{R}$$



Thm: ker A is a subspace of $\mathbb{R}^n$.

Pf: ① For any $v, w \in \ker A$.

$$Av = 0, \quad Aw = 0$$

$$A(v+w) = Av + Aw = 0$$

so $v+w \in \ker A$.

② For any $v \in \ker A$, $c \in \mathbb{R}$,

$A(cv) = c \cdot (Av) = 0$

so $cv \in \ker A$.

Application of subspaces to ODE.

Ex: $\{f \mid f'' + f = 0\} = W$

Find solutions $\sin x$, $\cos x$.

then any linear combination

$c_1 \sin x + c_2 \cos x$ is a solution

because W is a subspace.

Span.

Defn: Given $v_1 \dots v_k \in V$,

the span of $v_1 \dots v_k$ denote by

$\text{span}(v_1 \dots v_k)$ is the set

$$\text{span}(v_1 \dots v_k) = \{c_1v_1 + c_2v_2 + c_3v_3 + \dots + c_kv_k \mid c_1 \dots c_k \in \mathbb{R}\}$$

$\underline{c_1v_1 + \dots + c_kv_k}$ linear combination of $v_1 \dots v_k$.

Thm: $\text{span}(v_1 \dots v_k)$ is a subspace of V.

Pf: ① $v = c_1v_1 + \dots c_kv_k \in V$

$w = d_1v_1 + \dots + d_kv_k \in V$

$v + w = (c_1 + d_1)v_1 + \dots + (c_k + d_k)v_k \in V$

② $\forall c \in \mathbb{R}$,

$c \cdot (c_1v_1 + \dots + c_kv_k) = (c \cdot c_1)v_1 + \dots + (c \cdot c_k)v_k \in V$

Question: Is $w \in \text{Span}(v_1, \dots, v_k)$?

Ex: Is $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$?

Does there exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

row reduce: $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & 2 \\ 3 & 6 & 0 \end{bmatrix}$

How about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 4 & x \\ 2 & 5 & y \\ 3 & 6 & z \end{array}\right] \to \begin{bmatrix} 1 & 4 & x \\ 0 & -3 & y-2x \\ 0 & 0 & z-2y+x \end{bmatrix}$$

has a solution if

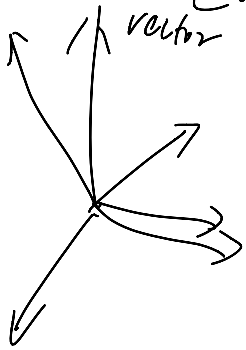
$z - 2y + x = 0$.

So $\operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right)$ is the plane $z - 2y + x = 0$.

Anothe approach:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \langle -3, 6, -3 \rangle$$

normal vector



$$-3x + 6y - 3z = 0$$

the same plane.

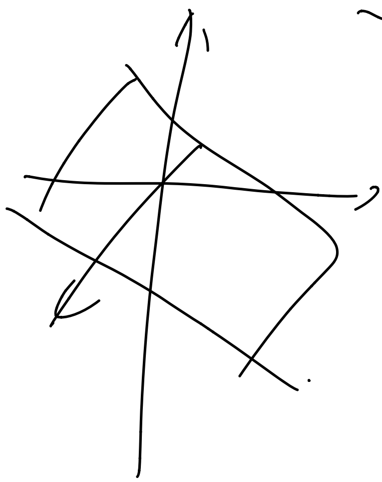
Rmk: ① $\text{span}(v_1, cv_1) = \text{span}(v_1)$

② $\text{span}\left(\underset{v_1}{\begin{bmatrix}1\\2\\3\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}4\\5\\6\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}7\\8\\9\end{bmatrix}} \right)$

$= \underbrace{\text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix} \right)}$.

2-dimensional plane.

$(v_3 = 2v_2 - v_1.)$



span of k-vectors does not have to be k-dimensional.