Linear independence/dependence, dimension.

Span.

Defn: Say a collection $\{v_1, \ldots v_k\}$ spans V if $\operatorname{span}(v_1 \ldots v_k) = V$

Spanning sets can contain redundant vectors.

e.g. $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ spans $\mathbb{R}^2$,

so does $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$. or $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$. or $\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$.

$\operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \mathbb{R}^2$ because.

$$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

How about $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$.

$$\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}, \quad \text{so}$$

$\begin{pmatrix}0\\1\end{pmatrix} \in \text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$.

So $\text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \subset \text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$

Hence $\text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\} = \mathbb{R}^2$.

Why?

$$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}.$$

$$= a\begin{pmatrix}1\\0\end{pmatrix} + b\left(\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= \underbrace{(a-b)\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}1\\1\end{pmatrix}}_{\text{linear combinations of } \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}}.$$

Want to consider spanning set with no redundancy.

Defn: A list $\{v_1 \cdots v_k\}$ in a vector space V is linearly independent if there exists $c_1, \cdots c_k \in \mathbb{R}$, not all zero, such that

$$(*) \quad c_1 v_1 + \cdots + c_k v_k = 0$$

If the only linear combination $c_1 v_1 + \cdots + c_k v_k = 0$ has solution $c_1 = c_2 = \cdots = c_k = 0$, say $\{v_1 \cdots v_k\}$ is linearly independent.

Why this is related to redundancy?

Say linear dependent,

$$c_1 v_1 + \cdots + c_k v_k = 0 \quad \text{and } c_1 \neq 0$$

then $v_1 = -\left(\frac{c_2}{c_1}\right) v_2 - \frac{c_3}{c_1} v_3 - \cdots - \frac{c_k}{c_1} v_k$

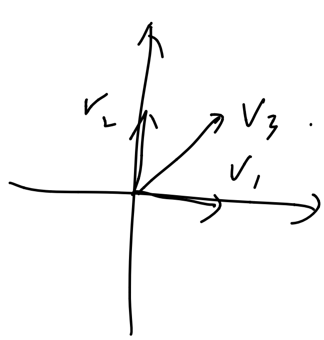
so $v_1 \in \text{span}(v_1, \cdots v_k)$, hence

v_1 is redundant.

In example $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$

$v_1 \quad v_2 \quad v_3$.

$$v_1 + v_2 - v_3 = 0.$$



Ex: Let $V = \{ f$ smooth function of x

$f'' + f = 0 \}$.

a) Is $\{\sin x, \cos x\}$ linearly independent?

Ans: Yes, suppose

$C_1 \sin x + C_2 \cos x = 0$ zero function.

Then $x = 0$, $C_1 \cdot 0 + C_2 \cdot 1 = 0 \Rightarrow C_2 = 0$

$x = \frac{\pi}{2}$, $C_1 \cdot 1 + C_2 \cdot 0 = 0 \Rightarrow C_1 = 0$

So $C_1 = C_2 = 0$.

b). Is $\{\underset{v_1}{\sin x}, \underset{v_2}{\cos x}, \underset{v_3}{\sin(x + \frac{\pi}{4})}\}$ linearly independent?

Ans: No. By trig identities

$$\sin(x+\tfrac{\pi}{4}) = \sin x \cos\tfrac{\pi}{4} + \cos x \sin\tfrac{\pi}{4}.$$

$$= \tfrac{\sqrt{2}}{2}\sin x + \tfrac{\sqrt{2}}{2}\cos x.$$

so $\frac{\sqrt{2}}{2} v_1 + \frac{\sqrt{2}}{2} v_2 - v_3 = 0.$

Facts: ① $\{v_1, v_2\}$ is linearly dependent

$\Leftrightarrow$ one vector is a scalar multiple of the other.

② Given column vectors

$$\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \cdots \begin{bmatrix} | \\ v_k \\ | \end{bmatrix} \in \mathbb{R}^n.$$

$\{v_1 \cdots v_k\}$ is linearly independent

$\Leftrightarrow$ $\ker A = \{0\}$, where $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$

This is because.

$$c_1 v_1 + \cdots + c_k v_k = A \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

Ex: Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

linearly independent?

Row reduce $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = A \longrightarrow$

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}.$$

one free variable.

So $\ker A \neq \{0\}$. Hence linearly dependent.

Ex: Is $\{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^3$ linear independent?

Ans: No!

Because $\begin{bmatrix} | & & | \\ v_1 & \cdots & v_4 \\ | & & | \end{bmatrix}$ has rank ≤ 3.

4 columns, at least one free variable.

Fact: Any collection of $> n$ vectors in $\mathbb{R}^n$ is linearly dependent.

A spanning set with no redundancy is called a basis.

Def: A set $\{v_1 \cdots v_k\} \subset V$ is a basis for V if

1) $\{v_1 \cdots v_k\}$ is linearly independent.
2) $\mathrm{span}(v_1 \cdots v_k) = V$.

Ex: In $\mathbb{R}^n$, $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$

is called the standard basis or coordinate basis.

notation: $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ ith row.

Example of basis.

① $\{1, x, x^2\}$ is a basis for

$P_2 = \{$ polynomials of degree $\leq 2\}$.

$a_0 + a_1 x + a_2 x^2 = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2$.

and $\{1, x, x^2\}$ is linearly independent.

② $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

is a basis for $M_{2\times 2}$.

Ex: $\left\{ \overset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$ is a basis for $\mathbb{R}^3$.

Why: ① $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ A invertible.

So $\ker A = \{0\} \Rightarrow \{v_1, v_2, v_3\}$ linearly indep.

② $A \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b$

always has a solution for any b.

Fact: Any two bases for the same vector space have the same number of elements.

Defn: The dimension of V is the number of elements in any basis.

If $\{v_1 \cdots v_k\}$ is a basis, then

3) $k = \dim V$.

1) $\{v_1 \cdots v_k\}$ is linearly independent.

2) $\operatorname{span}(v_1 \cdots v_k) = V$.

Any two of 1), 2), 3) implies

$\{v_1, \cdots v_k\}$ is a basis.

Why different basis?

Ex: (Partial fractions).

$$V = \left\{ \text{all rational functions of form } \frac{ax^2+bx+c}{(x-1)(x-2)(x-3)} : a, b, c \in \mathbb{R} \right\}.$$

V has basis

$$B = \left\{ \frac{1}{(x-1)(x-2)(x-3)}, \frac{x}{(x-1)(x-2)(x-3)}, \frac{x^2}{(x-1)(x-2)(x-3)} \right\}$$

So $\dim V = 3$.

Claim $C = \left\{ \frac{1}{x-1}, \frac{1}{x-2}, \frac{1}{x-3} \right\}$, is also a basis.

why? 3) C has 3 elements $= \dim V$.

1) C is linearly independent

because if $\frac{c_1}{x-1} + \frac{c_2}{x-2} + \frac{c_3}{x-3} = 0$.

let $x \to 1$, $\frac{1}{x-1} \to \infty$.

$\frac{1}{x-2} \to -1$ So $c_1 = 0$.

$\frac{1}{x-3} \to \frac{1}{-2}$

Let $x \to 2$, $\quad c_2 = 0$.

$x \to 3$, $\quad c_3 = 0$.

3) + 1) $\Rightarrow$ C is a basis

C is a better basis for integration.

$$\int \frac{1}{x-1}\, dx = \log|x-1| + C.$$

Need to know how to write

$$\frac{ax^2+bx+c}{(x-1)(x-2)(x-3)}$$ as linear combinations

of $\left\{ \frac{1}{x-1}, \frac{1}{x-2}, \frac{1}{x-3} \right\}$ to find

$$\int \frac{ax^2+bx+c}{(x-1)(x-2)(x-3)}\, dx.$$