1. $\hat{u}(w, y)$

$$(-iw)^2 \hat{u}(w,y) + \hat{u}_{yy} = 0$$

$$\hat{u}_{yy} = w^2 \hat{u}(w,y)$$

$$\hat{u} = A(w) e^{-wy} + B(w) e^{wy}$$

since $\lim_{x^2+y^2 \to \infty} u(x,y) = 0$

$$\hat{u}(w,y) = A(w) \cdot e^{-|w|y}$$

$$u(x,0) = \frac{2}{x^2+4}.$$

So $\hat{u}(w,0) = \widehat{\frac{2}{x^2+4}} = \frac{1}{2} e^{-|w| \cdot 2}$

Check the FT table

$$\hat{u}(w,y) = \frac{1}{2} e^{-|w| \cdot 2} \cdot e^{-|w| \cdot y} = \frac{1}{2} e^{-|w|(y+2)}$$

$$u(x,y) = \frac{(y+2)}{x^2+(y+2)^2}.$$

2. Poisson equation.

① First find solution to

$$\Delta u_0 = 4r^{-4}$$

Assume $u_0 = u_0(r)$. (since only depends on r)

$\frac{1}{r}(r u_0')' = 4r^{-4}$,

$r u_0'(r) = -2r^{-2} + C_1$

$u_0'(r) = -2r^{-3} + \frac{C_1}{r}$

$u_0(r) = r^{-2} + C_1 \ln r + C_2$.

$u_0(r)$ bounded $\Rightarrow C_1 = 0$.

We can make $u_0(1) = 0$ by $C_2 = -1$.

$u_0(r) = r^{-2} - 1$.

② Find solution to $w = u - u_0$.

$$\Delta w = 0$$
$$w(1, \theta) = 3\cos 3\theta - \sin 4\theta.$$

General solution for Laplace equation in 2D.

$$w(x, y) = A_0 + A_0' \ln r + \sum_{n=1}^{+\infty} A_i r^n \cos n\theta + \sum_{n=1}^{+\infty} B_i r^n \sin n\theta + \sum_{n=1}^{+\infty} C_i r^{-n} \cos n\theta + \sum_{n=1}^{+\infty} D_i r^{-n} \sin n\theta$$

(Got this by separation of variables $u(r, \theta) = \phi(\theta) \cdot G(r)$).

Since w bounded. no $\ln r$, or r^n,

$$w(x,y) = 3 r^{-3} \cos 3\theta - r^{-4} \sin 4\theta.$$

$$u(x,y) = r^{-2} - 1 + 3 r^{-3} \cos 3\theta - r^{-4} \sin 4\theta.$$

3. a) Multiply the equation by

$$e^{\int \frac{2}{x}} = e^{2 \cdot \ln x} = (e^{\ln x})^2 = x^2$$

so $x^2 u'' + 2x u' - 6 + \lambda x^2 u$

$= 0$

$$(x^2 u')' - 6 + \lambda x^2 u = 0.$$

$p = x^2$, $q = -6$, $r = x^2$.

b) Notice that this is not regular because

$$p(0) = r(0) = 0$$

we have learned two types of irregular equations

Bessel: $x^2 u'' + x u' - n^2 + \lambda x^2 u = 0.$

irregular at $x = 0$.

Legendre: $\frac{d}{dx}((1-x^2) g')' + \mu - \frac{n^2}{1-x^2} g = 0$

irregular at $x = 1, -1$, two end points.

So $x^2 u'' + 2x u' - 6 + \lambda x^2 u$

$= 0$

should be related to Bessel.

Comparing the equations on formula sheet.

It is spherical Bessel function appearing in solving Laplace eigenvalue problem in 3D spherical coordinate.

$$(\rho^2 f')' + \lambda \rho^2 - n(n+1) f = 0.$$

So solution is $j_2(\sqrt{\lambda}\rho) = \dfrac{\sqrt{\frac{\pi}{2}}\, \rho^{-\frac{1}{2}}}{\sqrt{\lambda}} J_{2+\frac{1}{2}}(\sqrt{\lambda}\rho)$

Let z_{nm} be the m-th positive zero of j_2'

or $J_{2+\frac{1}{2}}$

Then $\sqrt{\lambda}\cdot 1 = z_{2m} \Rightarrow \lambda = \left(\frac{z_{2m}}{1}\right)^2 = z_{2m}^{\ 2}$

$m = 1, 2, \ldots$

4. general solution to Laplace equation is

$$\sum_{n=0}^{+\infty}\sum_{m=0}^{n} A_{mn}\rho^n \, P_n^m(\cos\phi)\cdot\cos m\theta$$

$$+ \sum_{n=0}^{+\infty}\sum_{m=1}^{n} B_{mn}\rho^n P_n^m(\cos\phi)\sin m\theta .$$

$\rho^n \leftarrow$ n gives the degree of the polynomial. See the explanation at the end.

$n=4$, we have 9 solutions, $m=0,1,2,3,4$

$\rho^4 P_4(\cos\phi)$

$\rho^4 P_4^1(\cos\phi)\cos\theta$

$\rho^4 P_4^1(\cos\phi)\sin\theta .$

$$\rho^4 \; P_4^2(\cos\phi) \cos 2\theta,$$
$$\rho^4 \; P_4^2(\cos\phi) \sin 2\theta,$$
$$\vdots$$

These are the nine polynomials

$$u_1 \; \cdots \; u_9 .$$

In order to see this .

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi .$$

$$\rho^4 P_4(\cos\phi) = \rho^4 \cdot \left(\frac{35}{8} \cos^4\phi - \frac{15}{4} \cos^2\phi + \frac{3}{8} \right)$$
$$= \frac{35}{8} z^4 - \frac{15}{4} z^2 \cdot \rho^2 + \frac{3}{8} \cdot \rho^4$$

$$= \frac{35}{8} z^4 - \frac{15}{4} z^2 \cdot (x^2+y^2+z^2) + \frac{3}{8}(x^2+y^2+z^2)^2$$

$$\rho^n \cdot P_n^m(\cos\phi) \cos m\theta$$

$$\downarrow$$

In the expansion.

$\rho^n P_n^m(\cos\phi)$ is the summation of terms

like: $\rho^n (\cos\phi)^{n-m-2k} \cdot \left(\sqrt{1-\cos^2\phi}\right)^m \cos m\theta$

$$= \rho^n (\cos\phi)^{n-m-2k} \cdot \sin^m\phi \, \underbrace{\cos m\theta}_{\text{deg } m \text{ polynomial of } \cos\theta,\ \sin\theta}.$$

so it is a deg-n homogeneous polynomial of x, y, z.

5. From orthogonal relations, we have

$$\sum_{i=1}^{+\infty} a_i^2 \int_0^1 (J_0(j_i x))^2 x \, dx$$

$$= \int_0^1 x^2 \cdot x^2 \cdot x \, dx.$$

(we are using $\vec{v} = \sum a_i \vec{w}_i$

If $\vec{w}_i$ are orthogonal, then

$\langle \vec{v}, \vec{v} \rangle = \sum a_i^2 \langle \vec{w}_i, \vec{w}_i \rangle$.)

Left hand side =

$$\sum_{i=1}^{+\infty} a_i^2 \; \frac{1}{2} \left(\underbrace{J_0(j_i)^2}_{= 0} + J_1(j_i)^2 \right)$$

0 because j_i are zeros

$$= \frac{1}{2} \sum_{i=1}^{+\infty} a_i^2 \, J_1(j_i)^2$$

Right hand side $= \int_0^1 x^5 \, dx$

$$= \frac{1}{6}.$$

So $\sum_{i=1}^{+\infty} a_i^2 J_1(j_i)^2 = \frac{1}{3}.$

---

7. a) $u_E(x)$. equilibrium solution.

$$u_E'' + x = 0$$

$$u_E' = -\frac{1}{2}x^2 + C_1$$

$$u_E = -\frac{1}{6}x^3 + C_1 x + C_2.$$

$u_E'(0) = 1, \quad u_E'(1) = \alpha.$

$\Rightarrow \quad C_1 = 1, \quad -\frac{1}{2} + C_1 = \alpha.$

$$\alpha = \frac{1}{2}.$$

b). $w(x,t) = u(x,t) - u_E(x)$

satisfies

$$\begin{cases} w_t = w_{xx} \\ w_x(0,t) = 0, \quad w_x(1,t) = 0 \\ w(x,0) = f(x) + \frac{1}{6}x^3 - x - C_2. \end{cases}$$

$$w(x,t) = \sum_{n=0}^{+\infty} A_n \cos n\pi x \, e^{-n^2\pi^2 t}$$

$$A_0 = \int_0^1 \left(f(x) + \frac{1}{6}x^3 - x - C_2\right) dx.$$

$$A_n = 2\int_0^1 \left(f(x) + \frac{1}{6}x^3 - x + C_2\right) \cos(n\pi x) \, dx$$

$$= 2\int_0^1 \left(f(x) + \frac{1}{6}x^3 - x\right) \cos n\pi x \, dx$$

$$u(x,t) = -\frac{1}{6}x^3 + x + \int_0^1 \left(f(x) + \frac{1}{6}x^3 - x\right) dx + \sum_{n=1}^{+\infty} A_n \cos n\pi x \, e^{-n^2\pi^2 t}$$