Penn
UNIVERSITY of PENNSYLVANIA

MATH 241 FINAL EXAM

DECEMBER 16, 2019

No book, paper or electronic device may be used, other than a hand-written note sheet at most $8.5'' \times 11''$ in size. Cell phones should be **in your bags** and **turned off**.

This examination consists of Eight (8) long-answer questions with 15 points each. Please show all your work. Merely displaying some formulas is not sufficient ground for receiving partial credits. Please **box your answers**.

NAME (PRINTED): ____

RECITATION TIME:

INSTRUCTOR: (circle one)

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My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

____
Your signature

1	2	3	4	5	6	7	8	Total

Table 1: Boundary value problems for $\phi''(x) = -\lambda\,\phi(x)$

Boundary conditions	$\phi(0)=0$ $\phi(L)=0$	$\phi'(0)=0$ $\phi'(L)=0$	$\phi(-L)=\phi(L)$ $\phi'(-L)=\phi'(L)$
Eigenvalues	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n=1,2,3,\ldots$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n=0,1,2,3,\ldots$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n=0,1,2,3,\ldots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$	$A_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x$ $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x$	$a_0 = \frac{1}{2L}\int_{-L}^L f(x)\,\mathrm{d}x$ $a_n = \frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x$ $b_n = \frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$

Table 2: Orthogonality relations for sines and cosines

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0 \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0 \\ L, & n=m=0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \end{cases}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \\ 2L, & n=m=0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = 0$$

A Partial Table of Integrals

$$\int_0^x u\cos nu\;du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n\neq 0$$

$$\int_0^x u\sin nu\;du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n\neq 0$$

$$\int_0^x e^{mu}\cos nu\;du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\;du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\;du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m\neq n$$

$$\int_0^x \cos nu\cos mu\;du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m\neq n$$

$$\int_0^x \sin nu\sin mu\;du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m\neq n$$

The Laplacian $\triangle_{\mathbb{R}^2} = \nabla^2_{\mathbb{R}^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ on the plane in polar coordinates (r,θ) is

$$\triangle_{\mathbb{R}^2} = \nabla^2_{\mathbb{R}^2} = r^{-1}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + r^{-2}\frac{\partial^2}{\partial\theta^2} = \frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + r^{-2}\frac{\partial^2}{\partial\theta^2}$$

The Laplacian in spherical coordinates (ρ,ϕ,θ) in $\mathbb{R}^3$, with $\rho\geq 0$, $0\leq\phi\leq\pi$, $0\leq\theta\leq 2\pi$ is

$$\begin{aligned}\triangle_{\mathbb{R}^3} = \nabla^2_{\mathbb{R}^3} &= \frac{1}{\rho^2\sin\phi}\Big[\frac{\partial}{\partial\rho}\Big(\rho^2\sin\phi\frac{\partial}{\partial\rho}\Big) + \frac{\partial}{\partial\phi}\Big(\sin\phi\frac{\partial}{\partial\phi}\Big) + \frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\Big)\Big] \\ &= \rho^{-2}\frac{\partial}{\partial\rho}\Big(\rho^2\frac{\partial}{\partial\rho}\Big) + \rho^{-2}\,\triangle_{S^2},\end{aligned}$$

where

$$\triangle_{S^2} = \frac{1}{\sin\phi}\Big[\frac{\partial}{\partial\phi}\Big(\sin\phi\frac{\partial}{\partial\phi}\Big) + \frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\Big)\Big]$$

is the Laplacian on the unit sphere S^2. Recall that the spherical coordinates (ρ,ϕ,θ) is related to the Cartesian coordinates (x,y,z) by

$$x = \rho\,\sin\phi\,\cos\theta,\quad y = \rho\,\sin\phi\,\sin\theta,\quad z = \rho\,\cos\phi;$$

where $0\leq\phi\leq\pi$ is the co-altitude, and $0\leq\theta\leq 2\pi$ is the azimuthal angle.

Some properties of Bessel functions

$$\begin{aligned} e^{z(t-t^{-1})} &= \sum_{n\in\mathbb{Z}} J_n(z)t^n, \\ e^{z(t+t^{-1})} &= \sum_{n\in\mathbb{Z}} I_n(z)t^n.\end{aligned}$$

Denote by $\mathscr{C}_\nu(z)$ any of the four Bessel functions $J_\nu(z), Y_\nu(z), H_\nu^{(1)}(z), H_\nu^{(1)}(z)$. We have

$$\mathscr{C}_{\nu-1}(z) + \mathscr{C}_{\nu+1}(z) = \frac{2\nu}{z}\,\mathscr{C}_\nu(z),$$

$$\begin{aligned} \mathscr{C}_{\nu-1}(z) - \mathscr{C}_{\nu+1}(z) &= 2\,\tfrac{d}{dz}\mathscr{C}_\nu(z) \quad \text{if } \nu \neq 0 \\ -\mathscr{C}_1(z) &= \tfrac{d}{dz}\mathscr{C}_0(z) \end{aligned}$$

$$\begin{aligned} \left(z^{-1}\tfrac{d}{dz}\right)\left[z^\nu \mathscr{C}_\nu(z)\right] &= z^{\nu-1}\mathscr{C}_{\nu-1}(z) \\ \left(z^{-1}\tfrac{d}{dz}\right)\left[z^{-\nu} \mathscr{C}_\nu(z)\right] &= -z^{-\nu-1}\mathscr{C}_{\nu+1}(z)\,. \end{aligned}$$

The functions $I_\nu(z), K_\nu(z)$ satisfy similar recurrence relations with different signs at places.

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z}\,I_\nu(z), \quad I_{\nu-1}(z) + I_{\nu+1}(z) = 2\,\tfrac{d}{dz}I_\nu(z),$$

$$\left(z^{-1}\tfrac{d}{dz}\right)\left[z^\nu I_\nu(z)\right] = z^{\nu-1}I_{\nu-1}(z), \quad \left(z^{-1}\tfrac{d}{dz}\right)\left[z^{-\nu} I_\nu(z)\right] = z^{-\nu-1}I_{\nu+1}(z)\,.$$

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z}\,K_\nu(z), \quad K_{\nu-1}(z) + K_{\nu+1}(z) = -2\,\tfrac{d}{dz}K_\nu(z),$$

$$\left(z^{-1}\tfrac{d}{dz}\right)\left[z^\nu K_\nu(z)\right] = -z^{\nu-1}K_{\nu-1}(z), \quad \left(z^{-1}\tfrac{d}{dz}\right)\left[z^{-\nu} K_\nu(z)\right] = -z^{-\nu-1}K_{\nu+1}(z)\,.$$

$$\mathscr{C}_{-n}(z) = (-1)^n\,\mathscr{C}_n(z)$$

for all $n \in \mathbb{Z}$, where $\mathscr{C}_n(z)$ denote any one of $J_n(z), Y_n(z), H_n^{(1)}(z), H_n^{(2)}(z)$.

When ν is fixed and $z \to \infty$, for every $\delta > 0$ we have

$$\begin{aligned} J_\nu(z) &= \sqrt{2/(\pi z)}\,\Big(\cos\left(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) + e^{|\mathrm{Im}(z)|}o(1)\Big), \\ Y_\nu(z) &= \sqrt{2/(\pi z)}\,\Big(\sin\left(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) + e^{|\mathrm{Im}(z)|}o(1)\Big), \\ H_\nu^{(1)}(z) &= \sqrt{2/(\pi z)}\,e^{\sqrt{-1}\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}\big(1 + o(1)\big), \\ H_\nu^{(2)}(z) &= \sqrt{2/(\pi z)}\,e^{-\sqrt{-1}\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}\big(1 + o(1)\big), \end{aligned}$$

uniformly for all $z \in \mathbb{C} \smallsetminus (-\infty, 0]$ with $|\mathrm{ph}(z)| \leq \pi - \delta$.

$$\begin{aligned} I_\nu(x) &= \tfrac{1}{2\pi x}\,e^x\,\big(1 + o(1)\big), \\ K_\nu(x) &= \tfrac{\pi}{2x}\,e^{-x}\,\big(1 + o(1)\big), \end{aligned} \quad \text{as } x \to \infty,\ x \in \mathbb{R}.$$

for every $\nu \geq 0$.

$$(\beta^2-\alpha^2)\int xJ_n(\alpha x)\,J_n(\beta x)\,dx = x\big[\alpha J_n'(\alpha x)\,J_n(\beta x) - \beta J_n'(\beta x)\,J_n(\alpha x)\big] + C \quad \forall n$$

$$\int xJ_n^2(\alpha x)\,dx = \frac{1}{2}\Big[x^2 J_n'(\alpha x)^2 + \Big(x^2 - \frac{n^2}{\alpha^2}\Big)\,J_n(\alpha x)^2\Big] + C \quad \forall n$$

Let $j_{n,1} < j_{n,2} < \cdots < j_{n,k} < \cdots$ be the positive zeros of $J_n(x)$, and let $j'_{n,1} < j'_{n,2} < \cdots < j'_{n,k} < \cdots$ be the positive zeros of $J_n'(x)$. Then

$$\int_0^1 xJ_n(j_{n,k}x)J_n(j_{n,l}x)\,dx = 0 = \int_0^1 xJ_n(j'_{n,k}x)J_n(j'_{n,l}x)\,dx \quad \text{if } k \neq l,$$

$$\int_0^1 xJ_n^2(j_{n,k}x)\,dx = \tfrac{1}{2}\,(J_n'(j_{n,k}))^2 = \tfrac{1}{2}\,J_{n+1}^2(j_{n,k}),$$

$$\int_0^1 xJ_n^2(j'_{n,k}x)\,dx = \tfrac{1}{2}\Big(1 - \tfrac{n^2}{(j'_{n,k})^2}\Big)\,J_n^2(j'_{n,k}).$$

Formulas Involving Bessel Functions

- Bessel's equation: $r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0$ – The only solutions of this which are bounded at $r=0$ are $R(r) = cJ_n(\alpha r)$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

$J_0(0)=1$, $J_n(0)=0$ if $n>0$. z_{nm} is the mth positive zero of $J_n(x)$.

- Orthogonality relations: If $m \neq k$, then

$$\int_0^1 xJ_n(z_{nm}x)J_n(z_{nk}x)\,dx = 0 \quad \text{and} \quad \int_0^1 x(J_n(z_{nm}x))^2\,dx = \frac{1}{2}J_{n+1}(z_{nm})^2.$$

- Recursion and differentiation formulas:

$$\frac{d}{dx}(x^nJ_n(x)) = x^nJ_{n-1}(x) \quad \text{or} \quad \int x^nJ_{n-1}(x)\,dx = x^nJ_n(x) + C \quad \text{for } n \geq 1 \quad (1)$$

$$\frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1}(x) \quad \text{for} \quad n \geq 0 \quad (2)$$

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x) \quad (3)$$

$$J_n'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x) \quad (4)$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad (5)$$

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \quad (6)$$

- Modified Bessel's equation: $r^2R'' + rR' - (\alpha^2r^2 + n^2)R = 0$ – The only solutions of this which are bounded at $r=0$ are $R(r) = cI_n(\alpha r)$.

$$I_n(x) = i^{-n}J_n(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

Associated Legendre and Spherical Bessel Functions

- Differential equation for associated Legendre Functions:

$$\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\left(\mu-\frac{m^2}{\sin\phi}\right)g=0.$$

  Using the substitution $x=\cos\phi$, this equation becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right)+\left(\mu-\frac{m^2}{1-x^2}\right)g=0.$$

  For each natural number m, this equation has non-zero solutions which are bounded on $[-1,1]$ only when $\mu=n(n+1)$ for some natural number $n\geq m$.

- Associated Legendre Functions:

$$P_m(x)=P_n^0(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ and } P_n^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x),$$

  $1\leq m\leq n$. Some examples are:

$$P_0(x)=1, \qquad P_1(x)=x,$$
$$P_2(x)=\tfrac{3}{2}x^2-\tfrac{1}{2}, \qquad P_3(x)=\tfrac{5}{2}x^3-\tfrac{3}{2}x,$$
$$P_4(x)=\tfrac{35}{8}x^4-\tfrac{15}{4}x^2+\tfrac{3}{8}, \qquad P_5(x)=\tfrac{63}{8}x^5-\tfrac{35}{4}x^3+\tfrac{15}{8}x.$$

- Orthogonality of Associated Legendre Functions: If n and k are both greater than or equal to m,

$$\text{If } n\neq k \text{ then } \int_{-1}^{1}P_n^m(x)P_k^m(x)dx=0 \text{ and } \int_{-1}^{1}\left(P_n^m(x)\right)^2dx=\frac{2(n+m)!}{(2n+1)(n-m)!}.$$

- Spherical Bessel Functions: $(\rho^2 f')'+(\alpha^2\rho^2-n(n+1))f=0$. If we define the spherical Bessel function $\mathbf{j}_n(\rho)=\sqrt{\pi/2}\,\rho^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\rho)$, then only solution of this ODE bounded near $\rho=0$ is $\mathbf{j}_n(\alpha\rho)$.

- Spherical Bessel Function Identity:

$$\mathbf{j}_n(x):=\sqrt{\pi/2}\,x^{-\frac{1}{2}}\,J_{n+\frac{1}{2}}(x)=x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right).$$

- Spherical Bessel Function Orthogonality: Let z_{nm} be the m-th positive zero of $\mathbf{j}_m$. If $m\neq k$, then

$$\int_0^1 x^2\,\mathbf{j}_n(z_{nm}x)\,\mathbf{j}_n(z_{nk}x)dx=0 \quad\text{and}\quad \int_0^1 x^2(\mathbf{j}_n(z_{nm}x))^2dx=\frac{1}{2}(\mathbf{j}_{n+1}(z_{nm}))^2.$$

One-Dimensional Fourier Transform

$$\mathcal{F}[u](\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}[U](x)=\int_{-\infty}^{\infty}U(\omega)e^{-i\omega x}d\omega$$

Table of Fourier Transform Pairs

Fourier Transform Pairs ($\alpha > 0$)		Fourier Transform Pairs ($\beta > 0$)	
$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$	$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\beta}} e^{-\frac{x^2}{4\beta}}$	$e^{-\beta\omega^2}$
$e^{-\alpha\|x\|}$	$\frac{1}{2\pi}\frac{2\alpha}{\omega^2+\alpha^2}$	$\frac{2\beta}{x^2+\beta^2}$	$e^{-\beta\|\omega\|}$
$u(x) = \begin{cases} 0 & \|x\| > \alpha \\ 1 & \|x\| < \alpha \end{cases}$	$\frac{1}{\pi}\frac{\sin\alpha\omega}{\omega}$	$2\frac{\sin\beta x}{x}$	$U(\omega) = \begin{cases} 0 & \|\omega\| > \beta \\ 1 & \|\omega\| < \beta \end{cases}$
$\delta(x - x_0)$	$\frac{1}{2\pi} e^{i\omega x_0}$	$e^{-i\omega_0 x}$	$\delta(\omega - \omega_0)$
$\frac{\partial u}{\partial t}$	$\frac{\partial U}{\partial t}$	$\frac{\partial^2 u}{\partial t^2}$	$\frac{\partial^2 U}{\partial t^2}$
$\frac{\partial u}{\partial x}$	$-i\omega U$	$\frac{\partial^2 u}{\partial x^2}$	$(-i\omega)^2 U$
xu	$-i\frac{\partial U}{\partial\omega}$	$x^2 u$	$(-i)^2\frac{\partial^2 U}{\partial\omega^2}$
$u(x - x_0)$	$e^{i\omega x_0} U$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)g(x-s)ds$	FG