PDEs

setting: seek to determine an unknown function $u(x,t)$ with multiple unknown variables giving eqns relating u and its partial derivatives.

Notation: $$u_t = \frac{\partial u}{\partial t}$$
$$u_x = \frac{\partial u}{\partial x}.$$
$$u_{xt} = \frac{\partial^2 u}{\partial t \partial x} = (u_x)_t.$$

Clairaut's Theorem:

$$u_{xt} = u_{tx}$$

$u(x,y,z)$

Laplacian: $$\Delta u = u_{xx} + u_{yy} + u_{zz}.$$

Examples of PDE.

Heat equation: $u_t = u_{xx} + u_{yy} + u_{zz} = \Delta u$.

Wave equation: $u_{tt} = \Delta u$.

Laplace equation: $\Delta u = 0$.

Differences from ODEs

- Derivatives with respect to different variables
- Generally do not solve initial value problems but boundary vaule problems

Ex: ODE. $u''(t) = 0$

$$u(t) = c_1 t + c_2.$$

C_1, C_2 are determined by

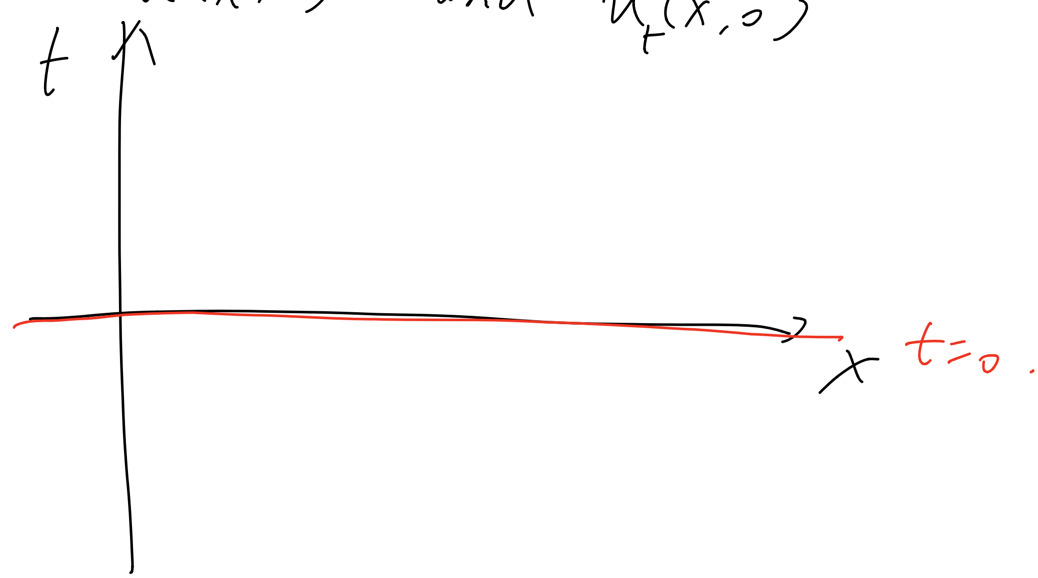
$u(0)$, $u'(0)$

PDE, $u(x,t)$. $u_{tt}=0$.

$u(x,t) = f(x)t + g(x)$.

$f(x)$, $g(x)$ are determined by

$u(x,0)$ and $u_t(x,0)$



Moral: · PDEs have undetermined functions.

· usually need boundary conditions to determine the solutions.

Prerequisites: - 1^{st}, 2^{nd} order linear ODEs

- Vector calculus (Divergence theorem)

- Linear algebra (linear transformations, bases)

1^{st} order linear ODE.

$$u' = 2u, \qquad u' - 2u = 0.$$

multiply by e^{-2t} (integration factor).

$$u' \cdot e^{-2t} - 2u \cdot e^{-2t} = 0$$

$$(u e^{-2t})' = 0$$

$$u = C \cdot e^{2t}.$$

General 1st order ODE.

$$u' + p(t)\,u(t) = 0.$$

multiply $e^{\int p(t)\,dt}$

$$\left(u \cdot e^{\int p(t)\,dt}\right)' = 0.$$

Nonhomogeneous :

$$u' + p(t)u = f(t).$$

$$tu' + 2u = t^2 - t.$$

$$u' + \frac{2}{t}u = t - 1.$$

mutiply $e^{\int \frac{2}{t}\,dt} = e^{2\log t} = t^2.$

$$t^2 u + 2tu = t^2(t-1)$$

$$(t^2u)' = t^3 - t^2$$

$$t^2 u = \frac{t^4}{4} - \frac{t^3}{3} + C.$$

$$u = \frac{t^2}{4} - \frac{t}{3} + \frac{C}{t^2}.$$

2nd order linear ODEs.

$$au'' + bu' + Cu = 0.$$

$$u = C_1 u_1(t) + C_2 u_2(t).$$

Example: $u'' - 9u = 0.$

char. poly. $r^2 - 9 = 0 \Rightarrow r = \pm 3$

$$u(t) = C_1 e^{3t} + C_2 e^{-3t}$$

If $u(0)=2$, $u'(0)=-1$.

$$\Rightarrow \begin{cases} c_1+c_2=2 \\ 3c_1-3c_2=-1. \end{cases} \Rightarrow \begin{array}{l} c_1=\frac{5}{6} \\ c_2=\frac{7}{6}. \end{array}$$

Notation:

$$\cosh x = \frac{e^x+e^{-x}}{2}.$$

$$\sinh x = \frac{e^x-e^{-x}}{2}.$$

Then $(\cosh x)' = \sinh x$

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$$(\cosh x)^2-(\sinh x)^2=1.$$

$\cosh 0=1$. $\quad (\cosh x)'\big|_{x=0}=0$

$\sinh 0=0$, $\quad (\sinh x)'\big|_{x=0}=1$.

$u(x)=c_3\cosh 3x+c_4\sinh 3x$.

$C_3 = u(0) = 2$

$3C_4 = u'(0) = -1 \Rightarrow C_4 = -\frac{1}{3}$

Ex: $u'' + 9u = 0$,

$r^2 + 9 = 0$, $r = \pm 3i$.

$u(x) = C_1 e^{3ix} + C_2 e^{-3ix}$.

Euler's identity

$$e^{a+bi} = e^a(\cos b + i \sin b).$$

$u(x) = C_1 \cos 3t + C_2 \sin 3t$

Vector calculus:

$f(x,y,z)$

$\nabla f = \langle f_x, f_y, f_z \rangle$. gradient vector field.

$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$.

$\text{div}\,\vec{F} = P_x + Q_y + R_z$. (divergence)

(source or sink of the flow)

$\Delta f = \nabla^2 f = \text{div}(\nabla f)$

$= f_{xx} + f_{yy} + f_{zz}$.

(Laplacian)

Divergence Theorem.

Ω bounded region

$\partial\Omega$ boundary surface

$\vec{F}$ vector field

$\vec{n}$ outward normal on $\partial\Omega$.

unit vectors.

$$\iiint_{\Omega} \operatorname{div} \vec{F} = \iint_{\partial\Omega} \langle \vec{F}, \vec{n} \rangle$$

$\langle \vec{F}, \vec{n} \rangle$ dot product.