Application of Heat equation.

(Question 6 in 1sth Midterm)

$\boxed{\text{Nuclear weapon. Fission weapon.}}$

The diffusion of particles (or chemical pollutant)

$u_t = k u_{xx}$. $\quad u(x,t)$ is the density.

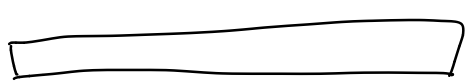
Nuclear chain reaction, $u(x,t)$ is the density of neutrons.

$$^{235}U + n \longrightarrow {}^{95}Sr + {}^{139}Xe + 2n + \underbrace{180\,MeV}_{\text{energy}}$$

neutrons

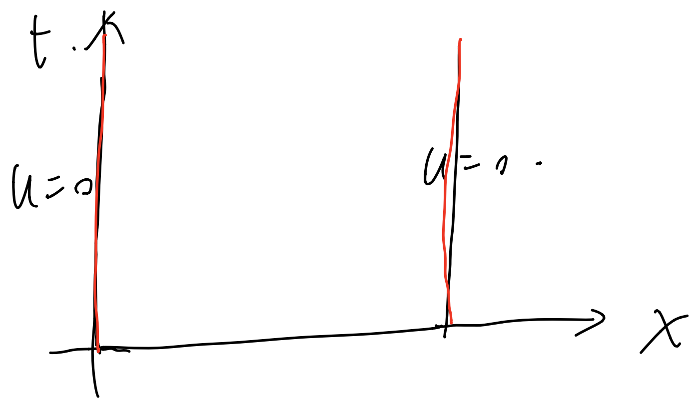
$$u_t = k u_{xx} + \underline{\beta u}. \qquad \beta > 0.$$

because chain reaction itself creates more and more neutrons.



^{235}Uranium.

$u(x,0) = 0$ $\qquad$ $u(x,L) = 0$.



Look for product sol'ns

$$u(x,t) = \phi(x) \cdot G(t)$$

$$\phi(x)\, G'(t) = K\, \phi''(x)\, G(t) + \beta\, \phi(x) \cdot G(t)$$

$$\frac{G'(t)}{G(t)} = K \frac{\phi''(x)}{\phi(x)} + \beta .$$

$$\frac{\phi''(x)}{\phi(x)} = -\lambda = \text{constant} .$$

$$\phi(0) = \phi(L) = 0 .$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 , \quad \phi_n(x) = \sin \frac{n\pi x}{L}$$

$$G'(t) = (-\lambda K + \beta)\, G(t)$$

$$G(t) = e^{\left(-\frac{n^2\pi^2 K}{L^2} + \beta\right)t}$$

$$u(x,t) = \sum \sin\frac{n\pi x}{L}\, e^{\left(-\frac{n^2\pi^2 K}{L^2} + \beta\right)t}.$$

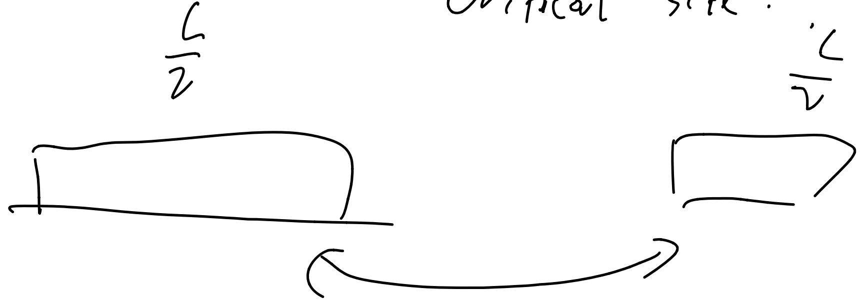
If we want $u(x,t)$ increases exponentially. (explosion)

then we need $-\frac{n^2\pi^2 K}{L^2} + \beta > 0$. for $\underline{n=1}$.

so $\underline{L > \sqrt{\frac{K}{\beta}}\,\pi}$.

$\downarrow$

critical size.



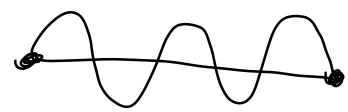
combine two to create explosion.

Wave equation:

$x=0$ $x=L$.

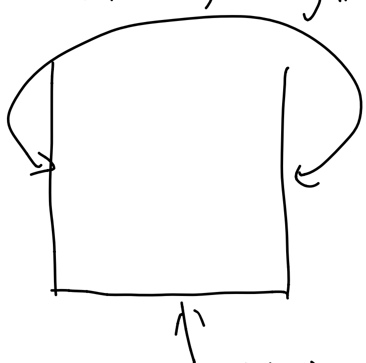
$$u_{tt} = c^2 u_{xx}$$

$u(0,t) = u(L,t) = 0$ BCs



$u(x,0) = f(x)$

$u_t(x,0) = g(x)$ ICs.



homogeneous

$u(x,0)$

$u_t(x,0)$

seperation of variables. $u(x,t) = \phi(x)\cdot G(t)$.

$$\phi(x)\cdot G''(t) = c^2\phi''(x)\cdot G(t)$$

$$\frac{\phi''(x)}{\phi(x)} = \frac{G''(t)}{c^2 G(t)} = -\lambda.$$

$$\phi''(x) = -\lambda\phi(x). \quad \phi(0) = \phi(L) = 0.$$

Boundary value problem $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

$$\phi_n(x) = \sin\frac{n\pi x}{L}.$$

$$G''(t) = -\frac{n^2\pi^2}{L^2}c^2 G(t).$$

So $G(t) = C_1 \cos\frac{n\pi c}{L}t + C_2 \sin\frac{n\pi c}{L}t.$

The sol'n to wave equation can be written as

$$u(x,t) = \sum_{n\geq 1}^{+\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi c}{L}t + \sum_{n=1}^{+\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi c}{L}t.$$

Matching initial conditions:

$$u(x,0) = \sum_{n=1}^{+\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$

$$u_t(x,0) = \sum_{n=1}^{+\infty} B_n \cdot \frac{n\pi c}{L}\sin\frac{n\pi x}{L} = g(x)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\frac{n\pi x}{L}\,dx.$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\cdot\sin\frac{n\pi x}{L}\,dx.$$