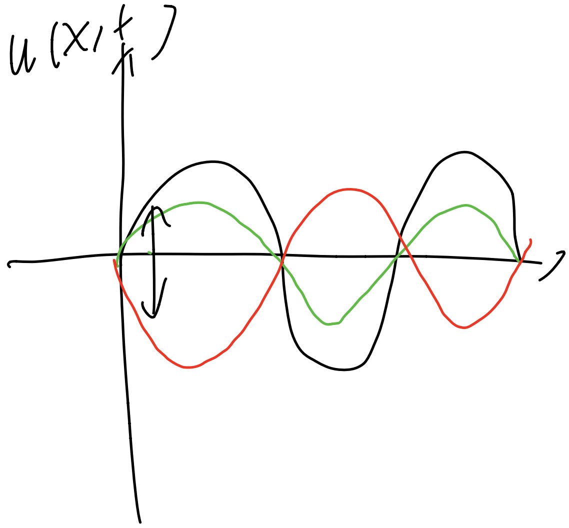
Normal mode:

$$\sin\frac{n\pi x}{L}\cdot\left(A_n\cos\frac{n\pi c}{L}t+B_n\sin\frac{n\pi c}{L}t\right)$$



something depending on time.

How to write a travelling wave.?

$$\sin\frac{n\pi x}{L}\cos\frac{n\pi c}{L}t$$

$$=\frac{1}{2}\left(\sin\frac{n\pi}{L}(x+ct)+\sin\frac{n\pi}{L}(x-ct)\right)$$

$$=F(x+ct)+G(x-ct)$$

$u(x,t) = F(x+ct)$ is the wave travelling from right to left.

solution to $u_t - c u_x = 0$. (transport equation)

(Read the solution to HW1)

$G(x-ct)$ is the wave travelling from left to right.

$$u_t + c u_x = 0$$

We want to find general solution to wave equation without any boundary conditions. (On the whole real line)

$$u_{tt} = c^2 u_{xx} \qquad -\infty < x < +\infty .$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

Inspired by solution to transport equation

we use the change of variables

$$X = x + ct \qquad x = \frac{X+Y}{2}$$

$$Y = x - ct . \qquad t = \frac{X-Y}{2c}$$

Calculate u_{XY}.

$$u_X = u_x \frac{1}{2} + u_t \cdot \frac{1}{2c}$$

$$(u_X)_Y = (u_X)_x \cdot \frac{1}{2} + (u_X)_t \cdot \left(-\frac{1}{2c}\right)$$

$$(u_X)_Y = \left(u_x \frac{1}{2} + u_t \cdot \frac{1}{2c}\right)_x \cdot \frac{1}{2}$$

$$+ \left(u_x \frac{1}{2} + u_t \cdot \frac{1}{2c}\right)_t \left(-\frac{1}{2c}\right)$$

$$= u_{xx} \cdot \frac{1}{4} - \frac{1}{4c^2} u_{tt}.$$

$$= \frac{1}{4c^2}\underbrace{(c^2 u_{xx} - u_{tt})}_{\text{from } u_{tt} = c^2 u_{xx}} = 0$$

so $u_{xy} = 0.$

$$u_x = F'(x)$$

$$u = \int F'(x) + G(y)$$

$$= F(x) + G(y)$$

$$= F(x+ct) + G(x-ct)$$

Plug in initial conditions:

$$u(x,0) = F(x) + G(x) = f(x)$$

$$u_t(x,0) = cF'(x) - cG'(x) = g(x)$$

Integrate: $F(x) - G(x) = \frac{1}{c}\left(\int_0^x g(s)\,ds + C_0\right)$

solve $F(x)$, $G(x)$

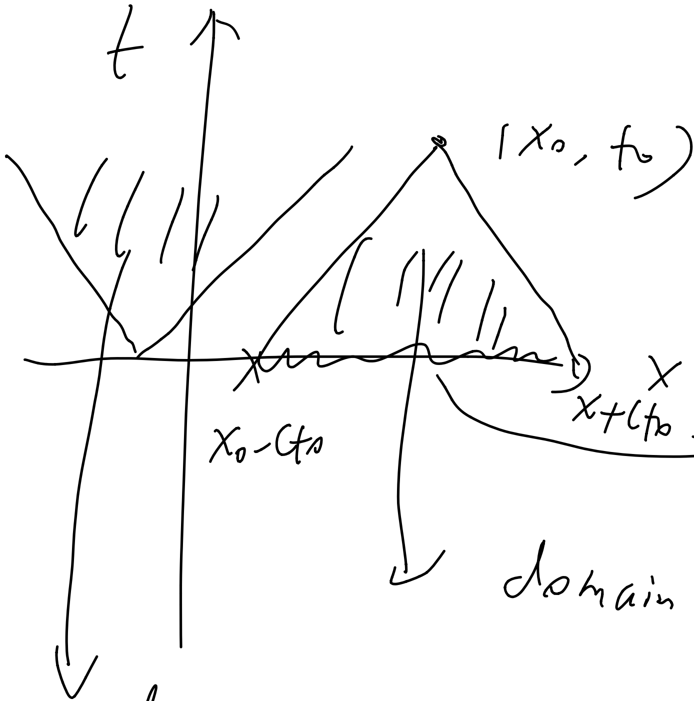
$$F(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\left(\int_0^x g(s)\,ds + C_0\right)\right)$$

$$G(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\left(\int_0^x g(s)\,ds + C_0\right)\right)$$

$$u(x,t) = F(x+ct) + G(x-ct)$$

d'Alembert's formula.

$$= \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$



value $u(x_0, t_0)$ depends on

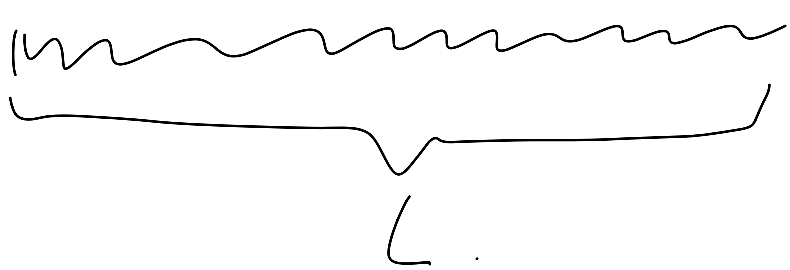
this part.

domain of dependence.

domain of influence.

Derivation of wave equation

elastic wave.

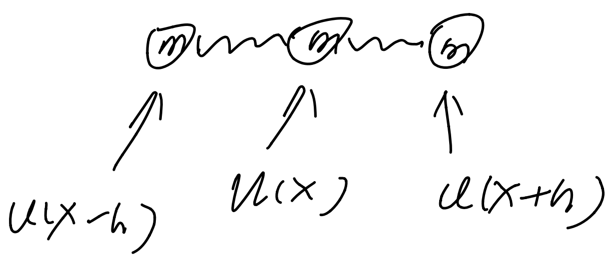


Newton's law $F = ma$.

Hooke's law $F = k \Delta x$

Divide 

into N parts. each part has length h, mass m.



$$F = m\, u_{tt}$$

$$F = k \cdot \left[\left(u(x+h,t) - u(x,t) \right) - \left(u(x,t) - u(x-h,t) \right) \right]$$

$$m \cdot u_{tt} = k \left[\left(u(x+h,t) - u(x,t) \right) - u(x,t) - u(x-h,t) \right]$$

$$h = \frac{L}{N}, \quad m = \frac{M}{N}, \quad K = k \cdot N.$$

so $$u_{tt} = \frac{KL^2}{M} \cdot \frac{1}{h^2}\Big(\big(u(x+h,t) - u(x,t)\big) - \big(u(x,t) - u(x-h,t)\big) \Big)$$

$$= \frac{KL^2}{M} u_{xx}$$

$$u_{tt} = c^2 u_{xx}.$$