Recall: Wave equation.

$$u_{tt} = c^2 u_{xx}. \qquad -\infty < x < +\infty$$

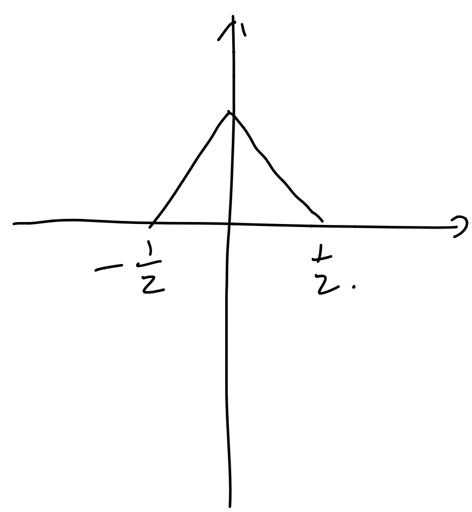
$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

D'Alembert's sol'n to wave eqn

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

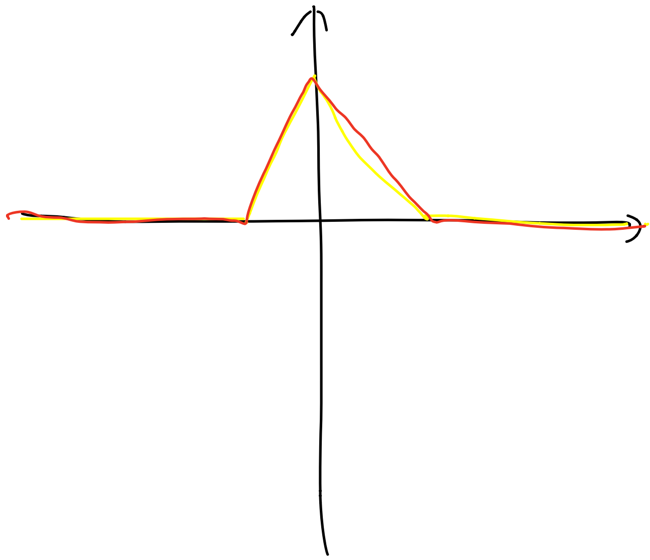
Example: $f(x) =$



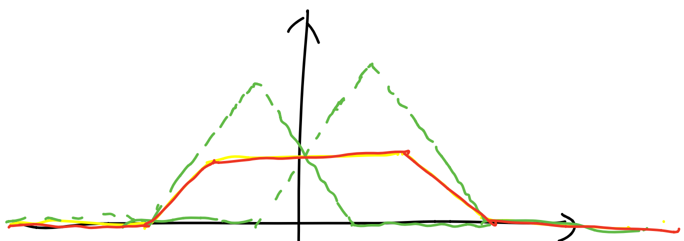
$g(x) \equiv 0.$

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$$

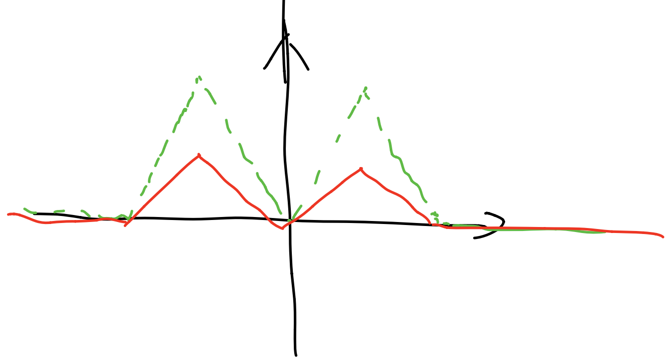
$t = 0$.



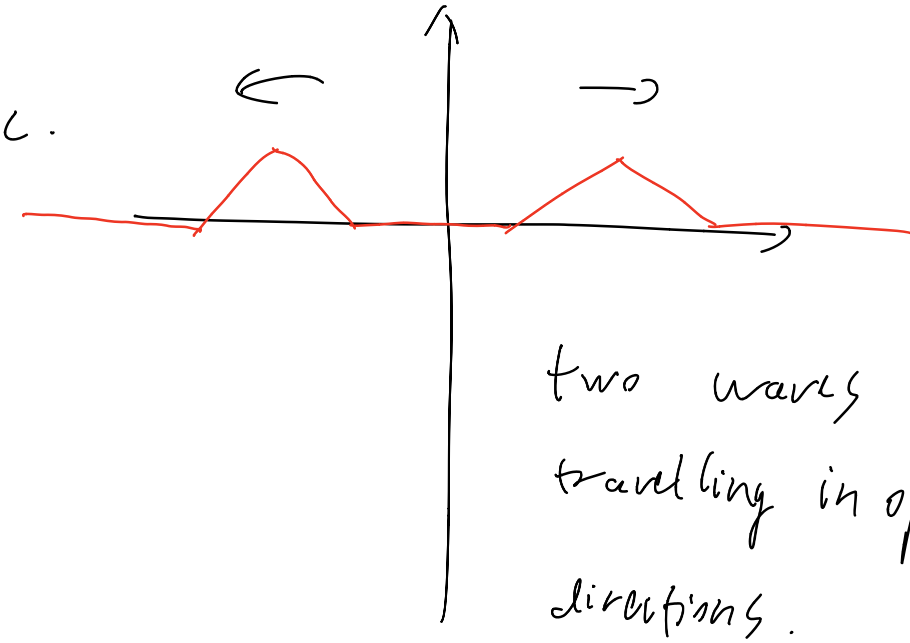
$t \ll 1$.



$t = \frac{1}{2c}$.



$t > \frac{1}{2c}$.



two waves travelling in opposite directions.

Domain of dependence:

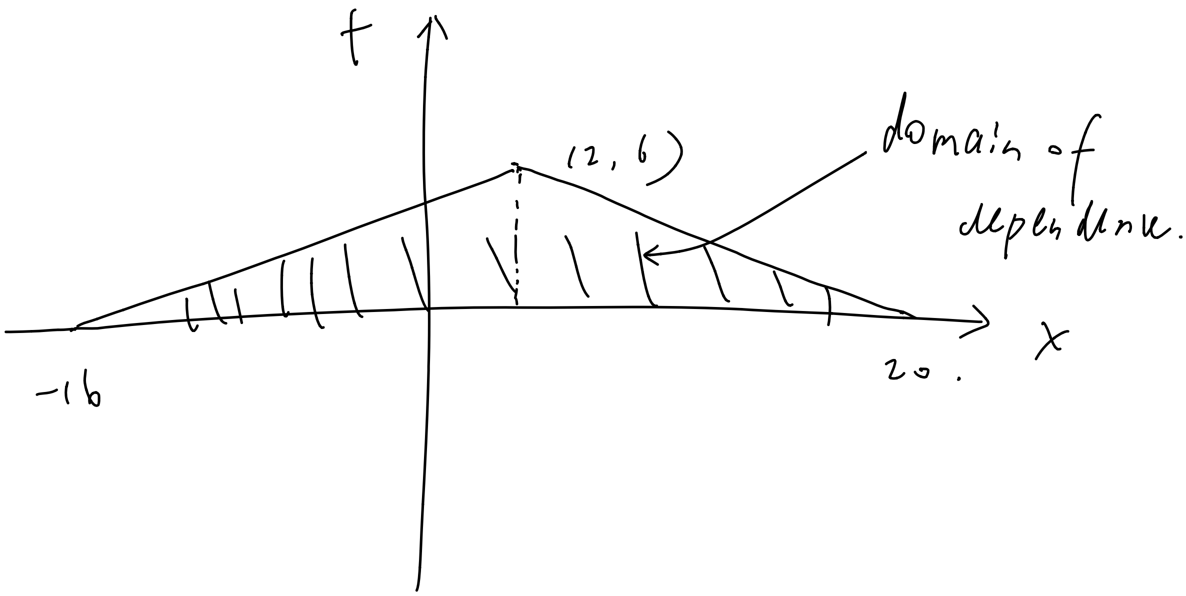
$u(x,t)$ solves $u_{tt} = 9u_{xx}$.

with $u(x,0)=f(x)$, $u_t(x,0)=g(x)$

Find the largest interval on which modifying $f(x)$ and $g(x)$ can change the value of $u(2,6)$

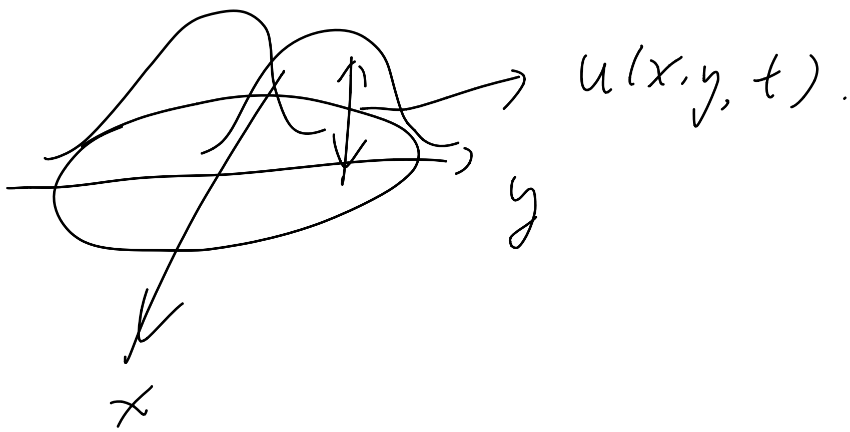
$$u(12,6) = \frac{1}{2}\left(f(20) + f(16)\right) + \frac{1}{6}\cdot\int_{-16}^{20} g(s)\,ds$$

so $u(12,6)$ depends only on values of $f(x)$, $g(x)$ in $[-16, 20]$



Higher dimensional wave eqn.

Vibrating membrane



wave eqn: $u_{tt} = c^2 \Delta u$

$$u(x,y,t)\Big|_{\partial D} = 0 \quad (BC)$$

Solve this using separation of variables

$$u(x,y,t) = \phi(x,y) \cdot G(t)$$

$$\Rightarrow \frac{\Delta \phi}{\phi} = \frac{G'(t)}{c^2 G(t)} = -\lambda.$$

$$\begin{cases} \Delta \phi = -\lambda \phi \\ \phi\big|_{\partial\Omega} = 0 \end{cases}$$ Boundary value problem.

(solve this later)

New topic Sturm-Liouville problem.

$$\begin{cases} \phi'' + \lambda \phi = 0. \\ \phi(0) = \phi(L) = 0. \end{cases}$$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n = \sin \frac{n\pi}{L} x$

orthogonality $\int_0^L \phi_n \phi_m \, dx = 0, \ m \neq n$

Ex: Heat eqn.

$$c\rho u_t = (K_0 u_x)_x + \alpha u.$$

separation of variables.

$$u(x,t) = \phi(x) G(t).$$

$$c\rho\, \phi(x) G'(t) = (K_0 \phi')' G(t) + \alpha \phi(x) G(t).$$

$$\Rightarrow \underbrace{\frac{G'(t)}{G(t)}}_{\text{depends on } t,} = \underbrace{\frac{1}{c\rho} \frac{(K_0 \phi')'}{\phi} + \frac{\alpha}{c\rho}}_{\text{depends on } x} = -\lambda.$$

$$\begin{cases} (K_0 \phi')' + \alpha \phi + \lambda c\rho \phi = 0. \\ \phi(0) = \phi(L) = 0. \end{cases}$$

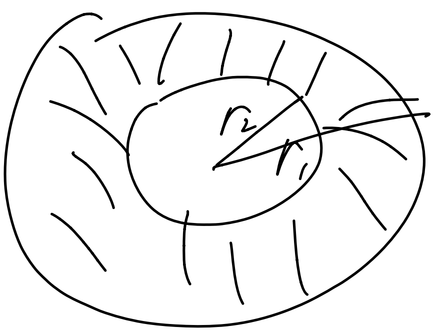
Boundary value problem.

Ex: Radial heat flow

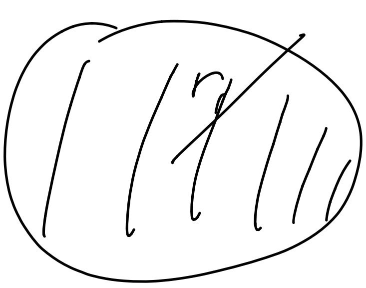
$$u(x,y,t) = \underbrace{u(r,\theta,t)}_{\text{not depending on } \theta}$$

$$u(r,t)$$

$$u_t = \Delta u = \frac{1}{r}(r u_r)_r$$



$$u(r_1,t)=0$$
$$u(r_2,t)=0.$$



$$u(r_1,t)=0$$
$$|u(0,t)|<+\infty.$$

Separation of variables.

$u(r,t) = \phi(r) \cdot G(t)$.

$\phi \cdot G'(t) = \frac{1}{r}(r\phi')' G(t)$

$\Rightarrow \quad \frac{(r\phi')'}{r\phi} = \frac{G'}{G} = -\lambda$.

$$\begin{cases} (r\phi')' + \lambda r \phi = 0. \\ \phi(r_1) = \phi(r_2) = 0. \end{cases}$$

The general form of a Sturm-Liouville eqn is

$(p(x)\phi')' + q\phi + \lambda \sigma \phi = 0. \quad a \le x \le b.$

$p(x)$, $q(x)$ $\sigma(x)$ are fcts of x.

$p(x) > 0, \quad r(x) > 0.$

BCs: $\alpha_1 \phi(a) + \beta_1 \phi'(a) = 0.$

$\alpha_2 \phi(b) + \beta_2 \phi'(b) = 0.$

Goal:
- Find eigenvalues λ_n.
- Find eigenfcts ϕ_n
- othogonality of eigenfcts.

Thm: ① all λ are real.

② $\exists$ eigenvalues

$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$

③ Each eigenspace is simple

i.e. 1-dim'l eigenspace

spanned by $\phi_n(x)$

④. $f(x) = \sum_{n=1}^{+\infty} a_n \phi_n(x)$

orthogonality. $\langle \phi_m, \phi_n \rangle = \int_a^b \phi_n \phi_m \sigma(x)\, dx = 0.$

if $m \neq n$.

$$a_n = \frac{\int_a^b f(x) \cdot \phi_n(x) \sigma(x)\, dx}{\int_a^b (\phi_n(x))^2 \sigma(x)\, dx}.$$