Recall: A regular Sturm-Liouville eigenvalue problem is an ODE. (assume $p(x)>0$, $\sigma(x)>0$)

$$(p(x)\phi')' + q(x)\cdot\phi(x) + \lambda\,\sigma(x)\phi(x) = 0$$

$x \in [a,b]$.

BCs: $\begin{cases} \alpha_1 \phi(a) + \beta_1 \phi'(a) = 0 \\ \alpha_2 \phi(b) + \beta_2 \phi'(b) = 0 \end{cases}$

Soon: An important thm.

(Existence, othogonality)

Q: Are all eigen-values positive for

$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = \phi(L) = 0 \end{cases}$?

Try to prove it by integration by parts.

Idea: Multiply both sides by ϕ and integrate.

$$\phi''\cdot\phi + \lambda\phi^2 = 0.$$

$$\int_0^L (\phi''\cdot\phi + \lambda\phi^2)\,dx = 0$$

$$\int_0^L \overset{g'}{\phi''}\cdot\overset{f.}{\phi}\,dx + \lambda\ \phi^2\,dx = 0.$$

Integration by parts.

$$-\int_0^L \phi'\cdot\phi'\,dx + \overset{0 \text{ by BCs}}{\boxed{\phi\cdot\phi'\Big|_0^L}}$$

$$+\lambda\int_0^L \phi^2\,dx = 0.$$

$$\Rightarrow \boxed{\lambda = \frac{\int_0^L (\phi')^2\,dx}{\int_0^L \phi^2\,dx}} \geq 0.$$

Rayleigh quotient

$\lambda = 0$? $\quad \phi' = 0, \Rightarrow \phi = \text{constant}$

$\phi(0) = \phi(L) = 0 \Rightarrow \phi = 0.$

Formula above solving λ in terms of ϕ is called Rayleigh quotient

---

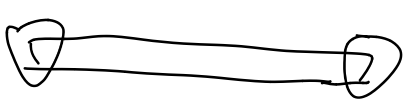
Same idea works for more complicated SL eval problems.

Ex: show λ is positive for:

$$\begin{cases} \dfrac{d}{dx}\left(x^{2019}\phi'\right) + \lambda e^{x}\phi(x) = 0 \\ \phi(1) = 0, \quad \phi'(2) = 0. \end{cases}$$

$1 \leq x \leq 2$

ice



insulated.

Mutiply both sides by ϕ, integrate.

$$\phi\left(x^{2019}\phi'\right)' + \lambda e^x \phi^2 = 0.$$

$$\underbrace{\int_1^2 \phi\cdot\left(x^{2019}\phi'\right)'}_{\parallel} + \lambda e^x\phi^2 = 0.$$

$$-\int_1^2 \phi' \, x^{2019}\phi' + \phi\cdot x^{2019}\phi'\Big|_1^2$$

$$+ \lambda\int_1^2 e^x\phi^2 = 0.$$

$$\boxed{\lambda = \frac{\int_1^2 x^{2019}(\phi')^2}{\int_1^2 e^x\phi^2}} \rightarrow \text{Rayleigh quotient.}$$

$\lambda = 0$! $\phi' = 0$, ϕ = constant

$\phi = 0$.

Rmk: It is possible to have negative λ if we have $q(x)\phi$ term

---

Thm: For regular S-L eval. problem,

(1) All λ are real

(2) There are infinitly many eigenvalues

$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \lambda_n < \cdots$

$\lambda_n \to +\infty$

(3) Each eigenvalue is <u>simple</u>

i.e. has 1-dimensional eigenspace, spanned by eigenfunction ϕ_n.

which has exactly $(n-1)$ zeros on (a,b).

(4). The set $\{\phi_n\}_{n=1}^{+\infty}$ is a complete othogonal set for piecewise smooth functions $f(x)$. with respect to.

$$f(x) \sim \sum_{n=1}^{+\infty} a_n \phi_n(x)$$

inner product with weight $\sigma(x)$

if $m \neq n$,

$$\langle \phi_n(x), \phi_m(x) \rangle_\sigma = \int_a^b \phi_n \cdot \phi_m \, \sigma \, dx.$$

$$= 0 .$$

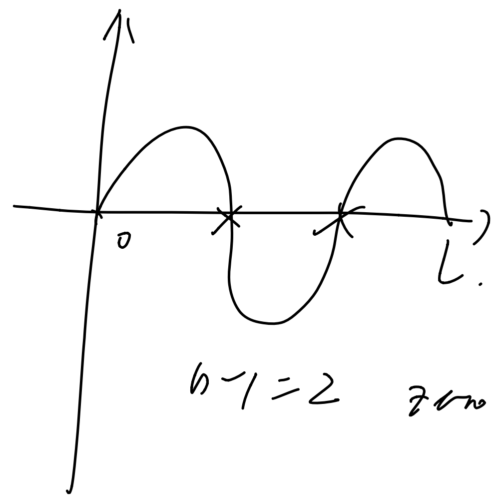
(5) The Rayleigh quotient relates λ to its eigenfunction ϕ:

$$\lambda = \frac{\left[-p \phi \phi'\right]_a^b + \int_a^b p(\phi')^2 - q\phi^2}{\int_a^b \phi^2 \sigma \, dx.}$$

Example:
$$\begin{cases} \phi'' + \lambda\phi = 0. \\ \phi(0) = \phi(L) = 0 \end{cases}$$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n = \sin\frac{n\pi x}{L}$.

$n=3$, $\sin\frac{3\pi x}{L}$



$n-1=2$ zeros

$f(x) = \sum a_n \phi_n.$

$$a_n = \frac{\langle f(x), \phi_n \rangle_\sigma}{\langle \phi_n, \phi_n \rangle_\sigma}$$

$$= \frac{\int_a^b f(x) \cdot \phi_n \, \sigma \, dx}{\int_a^b (\phi_n)^2 \, \sigma \, dx}.$$

Ex:

Solve heat equation

$$\begin{cases} e^x u_t = \frac{\partial}{\partial x}\left(x^{2019} u_x\right). \\ u(1,t) = u(2,t) = 0 \quad BC_s. \end{cases}$$

$$u(x,t) = \phi(x)\, G(t).$$

$$G'(t) = -\lambda G(t)$$

$$\left(x^{2019} \phi'\right)' + \lambda e^x \phi = 0.$$

$$\phi(1) = \phi(2) = 0.$$

$$u(x,t) = \sum_{n=1}^{+\infty} a_n \phi_n(x) e^{-\lambda_n t}.$$

$$u(0,t) = \sum_{n=1}^{+\infty} a_n \phi_n(x) = f(x)$$

$$c_n = \frac{\int_1^2 f(x)\cdot \phi_n e^x\, dx}{\int_1^2 \phi_n^2 e^x\, dx}. \quad (\sigma(x) = e^x)$$

Ex: Write the ODE in S-L form.

$$x^2\phi'' + 3x\phi' + \lambda\phi = 0.$$

$$\phi'' + \frac{3}{x}\phi' + \frac{\lambda}{x^2}\phi = 0$$

Use integrating factor

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$x^3\phi'' + 3x^2\phi' + \lambda \cdot x\phi = 0.$$

$$(x^3\phi')' + \lambda x\phi = 0.$$

$p(x) = x^3$, $q(x) = 0$, $\sigma(x) = x$.