regular S-L problem.

$$(p(x)\phi')' + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0.$$

$p(x) > 0, \quad \sigma(x) > 0.$

$$\begin{cases} \alpha_1 \phi(a) + \beta_1 \phi'(a) = 0 \\ \alpha_2 \phi(b) + \beta_2 \phi'(b) = 0 \end{cases} \quad \text{BCs.}$$

Why do we get a sequence of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$

and eigenfunctions $\phi_1, \phi_2, \cdots$

ϕ_n has $(n-1)$ zeros on (a, b)

(orthogonality) $\int_a^b \phi_n \phi_m \sigma(x)\, dx = 0.$

Recall Linear algebra.

(Symmetric matrices / spectral thm)

Fact: eigenvectors v, w of a symmetric matrix A with different eigenvalues are orthogonal.

$A = A^T$ $\qquad$ A^T transpose of A.

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$|\lambda I - A| = \begin{bmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{bmatrix}$$

$$= \lambda^2 - 4\lambda + 4 - 1$$

$$= \lambda^2 - 4\lambda + 3$$

$$= (\lambda-3)(\lambda-1)$$

$\lambda_1 = 3,$ $\quad A\vec{x} = \lambda \vec{x}.$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

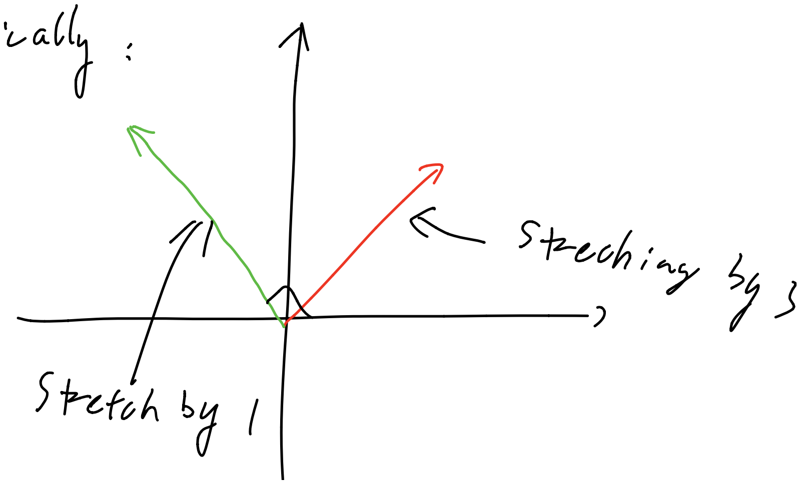
$$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = 1.$ $\quad A\vec{x} = \lambda \vec{x}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Geometrically :



Prop: If $A = A^T$, $A\vec{v} = \lambda_1 \vec{v}$, $A\vec{w} = \lambda_2 \vec{w}$, then inner product $\langle \vec{v}, \vec{w} \rangle = 0$.

Pf: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ $\quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \cdot \vec{w}$$

Consider $\langle A \cdot \vec{v}, \vec{w} \rangle = (A\vec{v})^T \cdot \vec{w}$

$$= \vec{v}^T \cdot A^T \cdot \vec{w}.$$

The same because $A = A^T$

$$\langle \vec{v}, A\vec{w} \rangle = (\vec{v}^T) \cdot A \cdot \vec{w}.$$

$A\vec{v} = \lambda_1 \vec{v}$, so

$$\lambda_1 \langle \vec{v}, \vec{w} \rangle = \langle \lambda_1 \vec{v}, \vec{w} \rangle = \langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A^T \vec{w} \rangle$$

$$= \lambda_2 \langle \vec{v}, \vec{w} \rangle.$$

so $\langle \vec{v}, \vec{w} \rangle = 0$.

Fact: $A = A^T$, then A is diagonalizable.

$\exists \underbrace{\vec{v}_1, \vec{v}_2, \vec{v}_3 \ldots \vec{v}_n}_{\text{basis}}$, such that

$A\vec{v}_i = \lambda_i \vec{v}_i$. $\lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots$

How to find λ_1.

Rayleigh quotient

$$\min \frac{\langle A\vec{v}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}.$$ (streching factor)

Rephrase S-L eqn

$$(p\phi')' + q\phi + \lambda r\phi = 0.$$

$$\Longleftrightarrow \quad L\phi = \lambda\phi, \quad \text{where}$$

$$L\phi := -\frac{1}{\sigma}\left[(p\phi')' + q\phi\right]$$

Use the inner product $\langle \, , \, \rangle_\sigma$

$$\langle f, g \rangle_\sigma = \int_a^b f(x)\, g(x)\, \sigma(x)\, dx$$

Goal: $\langle L\phi, \psi \rangle_\sigma = \langle \phi, L\psi \rangle_\sigma$

Pf: Integration by parts:

$$\langle L\phi, \psi \rangle_\sigma = \int_a^b -\frac{1}{\sigma}\left[(p\phi')' + q\phi\right] \psi \sigma \, dx$$

$$= -\int_a^b (p\phi')'\psi \, dx - \int_a^b q\phi\psi \, dx$$

$$\overset{\text{IBP}}{=} -\left[p\phi'\psi\right]\Big|_a^b + \int p\phi'\psi' dx - \int q\phi\psi \, dx.$$

$$\overset{\text{IBP}}{=} \overbrace{-\left[p\phi'\psi\right]\Big|_a^b + \left[p\phi\psi'\right]\Big|_a^b}^{\text{"0" by BCs}} - \int_a^b \phi(p\psi')' - \int_a^b q\phi\psi \, dx.$$

$= \langle \phi, L\psi \rangle_\sigma .$

Important question:

How to find λ_n, ϕ_n ?

Start from λ_1.

Rayleigh's quotient:

$$L\phi = \lambda \phi ,$$

Then
$$\lambda = \frac{\langle L\phi, \phi \rangle_\sigma}{\langle \phi, \phi \rangle_\sigma} .$$

$$= \frac{-p\phi\phi'\Big|_a^b + \int_a^b p(\phi')^2 - q\phi^2}{\int_a^b \phi^2 \sigma \, dx}$$

$$\lambda_1 = \min_{\substack{\phi \text{ satisfying} \\ \text{BCs}}} \frac{\langle L\phi, \phi \rangle_\sigma}{\langle \phi, \phi \rangle_\sigma}$$

Ex: Estimate λ_1 for

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = \phi(1) = 0. \end{cases}$$

$\lambda_1 = \pi^2$

Try $\phi = x(1-x)$

$$\lambda_1 \leq \frac{\int_0^1 (\phi')^2 dx}{\int_0^1 \phi^2 dx}$$

$$= \frac{\int_0^1 (1-2x)^2 dx}{\int_0^1 x^2(1-x)^2 dx}$$

$$= 10.$$

$\lambda_n \to +\infty$, corresponds to quantum mechanics in systems w/ very large energy.

Fact: asymptotically

$$\lambda_n \sim \left(\frac{n\pi}{\int_0^L \sqrt{\frac{r}{p}}\, dx} \right)^2$$

ϕ_n fluctuates more rapidly as $n \to \infty$

That's why ϕ_n has $(n-1)$ zeros.