Higher dim PDE.

$\Omega \subset \mathbb{R}^2$, $\Omega \subset \mathbb{R}^3$

regions in 2D or 3D.

Heat eqn. $u_t = \Delta u = u_{xx} + u_{yy} + u_{zz}$

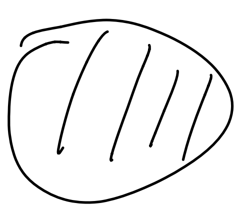
$u(x,y,z,t)$

Wave eqn. $u_{tt} = \Delta u = u_{xx} + u_{yy} + u_{zz}$

$u(x,y,z,t)$.

Seperation of variables.

Ex: Heat eqn on 2D domain



$$\begin{cases} u_t = \Delta u. \\ u|_{\partial\Omega} = 0. \end{cases}$$

$$u(x,y,t) = \phi(x,y)\cdot G(t)$$

$$\phi\cdot G' = \Delta\phi\cdot G$$

$$\frac{\Delta\phi}{\phi} = \frac{G'}{G} = -\lambda \cdot \text{ constant.}$$

$$G' = -\lambda G \Rightarrow G = e^{-\lambda t}.$$

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi|_{\partial\Omega} = 0. \end{cases}$$ eigenvalue problem.

Prop: $\lambda > 0$

Pf: Multiply by ϕ, integrate by parts

$$\underline{\iint_\Omega \Delta\phi\cdot\phi} = -\lambda\iint_\Omega \phi^2$$

$$= \boxed{\int_{\partial\Omega} \langle\nabla\phi, \vec{n}\rangle\phi}^{=0 \ (BC)} - \iint_\Omega |\nabla\phi|^2.$$

$$\Rightarrow \quad \lambda = \frac{\iint_\Omega |\nabla\phi|^2}{\iint_\Omega \phi^2} \geq 0.$$

If $\lambda = 0$, then $|\nabla\phi| = 0$.

$\phi = $ constant, $\phi|_{\partial\Omega} = 0$. $\phi = 0$.

So $\lambda > 0$.

Wave equation: (Vibrating membrane with fixed bdy).

$$u_{tt} = \Delta u.$$

$$u(x,y,t) = \phi(x,y)\, G(t).$$

$$G''(t)\,\phi = G \cdot \Delta\phi.$$

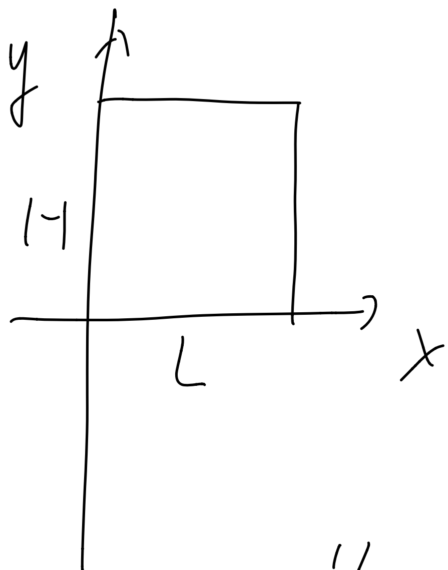
$$\frac{G''}{G} = \frac{\Delta\phi}{\phi} = -\lambda.$$

$$G'' = -\lambda G$$

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi|_{\partial\Omega} = 0 \end{cases} \Rightarrow \lambda > 0$$

$$G = C_1 \cos\sqrt{\lambda}\,t + C_2 \sin\sqrt{\lambda}\,t.$$

How to solve $\begin{cases}\Delta\phi+\lambda\phi=0\\ \phi|_{\partial\Omega}=0\end{cases}$



$\Omega=[0,L]\times[0,H]$.

$$\phi(x,y)=X(x)Y(y)$$

$$X''Y+XY''+\lambda XY=0.$$

$$\frac{X''}{X}+\frac{Y''}{Y}+\lambda=0$$

$$\frac{X''}{X}=-\lambda-\frac{Y''}{Y}=-\mu$$

$$\begin{cases} X'' = -\mu X \\ X(0) = X(L) = 0 \end{cases} \qquad (*)$$

$$\begin{cases} Y'' + (\lambda - \mu) Y = 0 . \\ Y(0) = Y(M) = 0 . \end{cases} \qquad (**)$$

$(*) \Rightarrow \mu_n = \left(\frac{n\pi}{L}\right)^2 , \; n = 1, 2 \ldots\ldots$

$$X_n(x) = \sin \frac{n\pi x}{L}$$

$(**) \Rightarrow (\lambda_{mn} - \mu_n) = \left(\frac{m\pi}{M}\right)^2$

$$Y_m(y) = \sin \frac{m\pi y}{M}$$

$m = 1, 2, \ldots$

$$\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2,$$

$m = 1, 2 \cdots -$

$n = 1, 2 - - -$

$$\phi_{mn} = \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y$$

$$\lambda_{11} = \left(\frac{\pi}{H}\right)^2 + \left(\frac{\pi}{L}\right)^2 \leftarrow \text{smallest.}$$

$$\lambda_{12} = \left(\frac{\pi}{H}\right)^2 + \left(\frac{2\pi}{L}\right)^2$$

$$\lambda_{21} = \left(\frac{2\pi}{H}\right)^2 + \left(\frac{\pi}{L}\right)^2$$

$\lambda_{22} \cdots \cdots$

Heat equ: $u(x,y,t)$

$$= \sum A_{mn} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \cdot e^{-\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]t}.$$

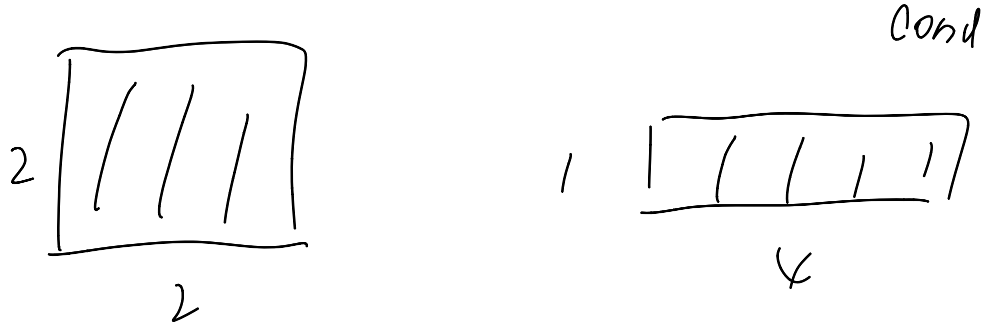
(Find A_{mn} by orthogonality)

$e^{-\lambda t}$ λ larger than

$e^{-\lambda t} \to 0$ more rapidly.

so the behaviour of heat eqn is dominated by λ_{11}.

Q: which shaped rectangle retains heat better (with Dirichlet conditions)

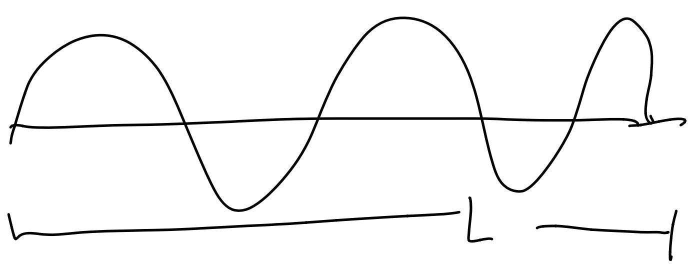


$$\lambda_{11} = \left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{2} \quad < \quad \lambda_{11} = \left(\frac{\pi}{4}\right)^2 + \pi^2 = \frac{17}{16}\pi^2$$

Better !

Can you hear the shape of a drum?

1D version $\quad u_{tt} = u_{xx}$.



$$u(x,t) = \left(\sin \frac{n\pi}{L} x\right) \cdot \left(C_1 \cos \frac{n\pi}{L} t + C_2 \sin \frac{n\pi}{L} t\right)$$

Frequency: How many periods during 1 unit time.

$$f = \frac{n\pi}{L} \Big/ 2\pi = \frac{n}{2L}$$

L large, f small.

Frequency of a drum:

$$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{W}\right)^2.$$

wave eqn solution:

$$\phi_{mn} \cdot \left(C_1 \cos \sqrt{\lambda_{mn}}\, t + C_2 \sin \sqrt{\lambda_{mn}}\, t\right)$$

$$f = \frac{\sqrt{\lambda_{mn}}}{2\pi}$$

for example $\lambda_{11} = \left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{H}\right)^2$

$$f = \frac{\sqrt{\left(\frac{\pi}{H}\right)^2 + \left(\frac{\pi}{L}\right)^2}}{2\pi}.$$