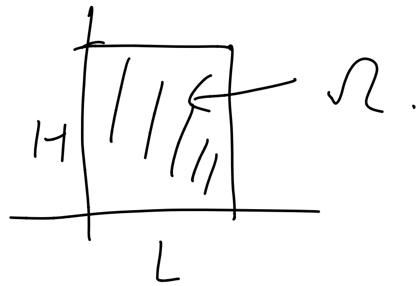
Recall:

$$\begin{cases} u_t = \Delta u, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$



seperation of variables:

$$\begin{cases} \Delta \phi + \lambda \phi = 0 \\ \phi = 0 \end{cases} \quad \text{on } \partial\Omega.$$

$$G'(t) = -\lambda G(t) \qquad G(t) = e^{-\lambda t}$$

$$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

$$\phi_{mn} = \sin\frac{n\pi}{L}x \, \sin\frac{m\pi}{H}y$$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y \, e^{-\lambda_{mn}t}$$

$u(x,y,0) = f(x,y)$. How to find A_{mn}.

(Orthogonality)

General eigen value problem:

$$\begin{cases} \Delta\phi + \lambda\phi = 0 \\ a\phi + b\langle\nabla\phi, \vec{n}\rangle = 0 \quad \text{on } \partial\Omega \end{cases}$$

Thm: ① All λ are real

② $\lambda_1 < \lambda_2 < \lambda_3 \cdots$

③ Eigenvalue need not to be <u>simple</u> one eigen vaule corresponds to a finite dimensional eigenspace

$\{\phi_n\}_{n=1}^{\infty}$ is a complete orthogonal basis

$f(x,y) = \sum a_n \phi_n(x,y)$

$\langle\phi_n, \phi_m\rangle = \iint \phi_n \phi_m \, dxdy = 0$ if $m \neq n$.

(must apply Gram-Schmidt to each eigenspace to get orthogonal basis)

(4). If $\Delta\phi = -\lambda\phi$,

then $$\lambda = \frac{-\int_{\partial\Omega} \phi \langle \nabla\phi, \vec{n} \rangle + \iint_{\Omega} |\nabla\phi|^2}{\iint_{\Omega} \phi^2}$$

If $H = L$, $\lambda_{12} = (\frac{\pi}{2L})^2 + (\frac{\pi}{L})^2$

$\lambda_{21} = (\frac{\pi}{L})^2 + (\frac{\pi}{2L})^2$.

$\lambda_{12} = \lambda_{21}$. $\phi_{12} = \sin\frac{\pi}{H}y \sin\frac{2\pi}{L}x$

$\phi_{21} = \sin\frac{2\pi}{H}y \sin\frac{\pi}{L}x$

Double Fourier series.

$$f(x,y) = \sum a_{nm} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y.$$

$$a_{nm} = \frac{\langle f, \phi_{nm} \rangle}{\langle \phi_{nm}, \phi_{nm} \rangle}$$

$$= \frac{4}{LH} \int_0^H \int_0^L f \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y \, dx\, dy$$

$$\left(\begin{array}{l} \text{Rayleigh quotient} \\ \text{multiply } \phi \text{ and integrate.} \end{array} \right)$$

Thm: Δ is self-adjoint.

$$\langle \Delta\phi, \psi \rangle = \langle \phi, \Delta\psi \rangle.$$

Pf:
$$\left(\begin{array}{l} a\phi + b\langle \nabla\phi, \vec{n} \rangle = 0 \\ a\psi + b\langle \nabla\psi, \vec{n} \rangle = 0 \end{array} \right) \text{ on } \partial\Omega$$

$$\phi \langle \nabla\psi, \vec{n} \rangle - \psi \langle \nabla\phi, \vec{n} \rangle = 0 \text{ on } \partial\Omega$$

$$\langle \Delta\phi, \psi \rangle = \iint (\nabla^2\phi)\psi$$

$$= \int_{\partial\Omega} \langle \nabla\phi, \vec{n} \rangle \cdot \psi - \iint_{\Omega} \langle \nabla\phi, \nabla\psi \rangle$$

$$\langle \phi, \Delta\psi \rangle = \int_{\partial\Omega} \langle \nabla\psi, \vec{n} \rangle \phi - \iint_{\Omega} \langle \nabla\psi, \nabla\phi \rangle$$

If ϕ_n, ϕ_m are two eigenfunctions

$$\Delta \phi_n = \lambda_n \phi_n \qquad \lambda_n \neq \lambda_m.$$
$$\Delta \phi_m = \lambda_m \phi_m$$

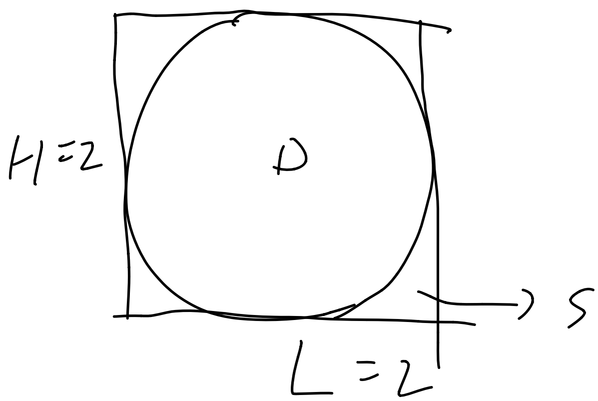
Then
$$\begin{aligned}
\lambda_n \langle \phi_n, \phi_m \rangle &= \langle \Delta \phi_n, \phi_m \rangle \\
&= \langle \phi_n, \Delta \phi_m \rangle \\
&= \lambda_m \langle \phi_n, \phi_m \rangle.
\end{aligned}$$

$$\langle \phi_n, \phi_m \rangle = 0$$

λ_1 is the minimal of the Rayleigh quotient.

$$\lambda_1 = \min_{\substack{\phi \text{ satisfy} \\ BC}} \frac{-\int \langle \nabla \phi, \vec{n} \rangle \phi + \iint |\nabla \phi|^2}{\iint \phi^2}$$

Example:



D disc of radius 1

S square of length 2.

D is contained in S.

Dirichlet Boundary.

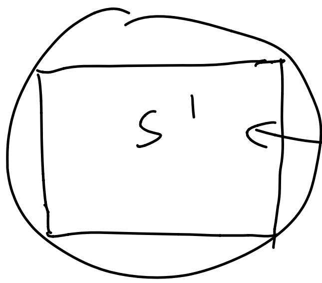
$$\lambda_1(D) = \min_{\phi|_{\partial D}=0} \frac{\iint |\nabla \phi|^2}{\iint |\phi|^2}$$

$$\geq \min_{\psi|_{\partial S}=0} \frac{\iint |\nabla \psi|^2}{\iint |\psi|^2}$$

$$= \lambda_1(S)$$

So $\lambda_1(D) \geq (\frac{\pi}{2})^2 + (\frac{\pi}{2})^2 = \frac{\pi^2}{2}$.

We can get upper bounds by

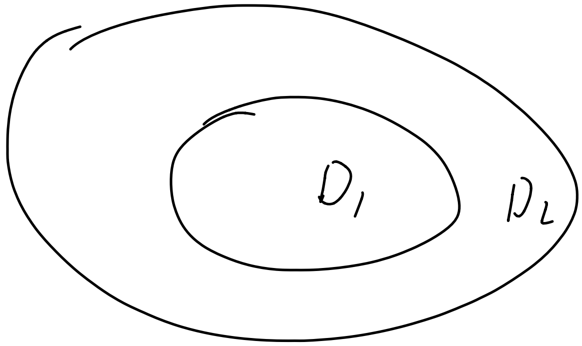
① 

put a square S' in D.

$\lambda_1(D) \le \lambda_1(S')$

② Test function $f(x,y) = 1 - r^2$.

$$\lambda_1(D) \le \frac{\iint |\nabla f|^2}{\iint f^2} \quad (f|_{\partial D} = 0)$$



Two drums.

D_1 is smaller than D_2 (contained in D_2)

Then frequency of D_1 $\frac{\lambda_1(D_1)}{2\pi}$ is higher than the frequency of D_2 $\frac{\lambda_1(D)}{2\pi}$.