Visualize nodal set.

$$\begin{cases} \Delta\phi + \lambda\phi = 0 \\ \phi|_{\partial\Omega} = 0 \end{cases}$$

λ corresponds to frequency.

The same frequency may corresponds to multiple eigen states.

solution to wave equation $u_{tt} = \Delta u$.

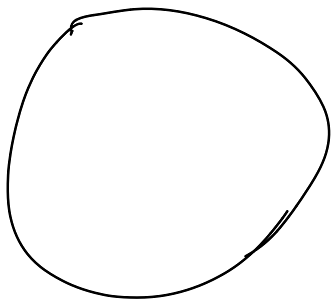
$$\phi_n(x,y) \cdot (C_1 \cos\sqrt{\lambda_n}\, t + C_2 \sin\sqrt{\lambda_n}\, t)$$

Nodal set (nodal curve in 2D)

is the set of zeros of $\phi_n(x,y)$

See the link of graphs, videos and codes on our websites for visualizing nodal sets using sand or salt on a drum.

Last time :



$$\lambda_1(D) \geq \frac{\pi^2}{2}.$$

$$\begin{cases} \Delta\phi + \lambda\phi = 0 \\ \phi|_{\partial D} = 0 \end{cases}$$

Let's use seperation of variables to solve

$$\phi(r,\theta) = f(r)\cdot g(\theta)$$

$$\begin{cases} \Delta\phi + \lambda\phi = 0. \\ \phi|_{\partial D} = 0. \end{cases}$$

$$\frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta} = -\lambda f\cdot g$$

$$\frac{1}{r}(rf'(r))' g(\theta) + \frac{1}{r^2} f(r) g''(\theta) = -\lambda f g$$

$$\frac{r(r f'(r))'}{f} + \frac{g''(\theta)}{g(\theta)} = -\lambda r^2.$$

$$\frac{r(rf')'}{f} + \lambda r^2 = -\frac{g''}{g} = \mu$$

μ is constant

$$\begin{cases} g'' = -\mu g \\ g(-\pi) = g(\pi) \\ g'(-\pi) = g'(\pi) \end{cases}$$

$$\mu_n = n^2, \quad g(\theta) = \begin{cases} \sin n\theta \\ \cos n\theta \end{cases}$$

Now we need to solve f.

$$\begin{cases} r(rf)' - n^2 f + \lambda r^2 f = 0. \\ 0 \le r \le R. \\ |f(0)| < +\infty, \quad f(R) = 0 \end{cases} \qquad (*)$$

S-L problem, but $\boxed{\text{not regular}}$.

$$p(r) = r, \quad q(r) = -\frac{n^2}{r}, \quad \sigma(r) = r$$

$$p(0) = \sigma(0) = 0, \quad q(0) \to -\infty.$$

Try to solve $(*)$

Analogue :
$$\begin{cases} \phi''(x) + \lambda \phi(x) = 0 \\ \phi(0) = \phi(L) = 0 \end{cases}$$

change of variable:

$$z = \sqrt{\lambda}\, x, \quad \text{then} \quad \frac{d\phi}{dx} = \frac{d\phi}{dz} \cdot \frac{dz}{dx} = \sqrt{\lambda} \frac{d\phi}{dz}$$

$$\frac{d^2\phi}{dx^2} = \frac{d\left(\sqrt{\lambda} \frac{d\phi}{dz}\right)}{dz} \cdot \frac{dz}{dx} = \lambda \frac{d^2\phi}{dz^2}$$

$$\phi''(x) + \lambda \phi(x) = 0$$

$$\|$$

$$\Downarrow$$

$$\frac{d^2\phi}{dz^2} + \phi(z) = 0. \leftarrow \text{equation not involving } \lambda.$$

$$\phi(z) = C_1 \cos(z) + C_2 \sin(z).$$

$$C_1 = 0 \leftarrow \phi(0) = 0$$

$$\sin(\sqrt{\lambda} L) = 0 \Rightarrow \sqrt{\lambda} L = n\pi$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

Back to (*):

$$r^2 f'' + r f' - n^2 + \lambda r^2 f = 0$$

Idea

1. change of variable to obtain some "standard" ODE (not involving λ)
2. Solve the "standard" ODE by inventing new names.

③ Study the behaviour of the new functions

④ Combine with boundary conditions to obtain the possible eigen values λ.

① $z = \sqrt{\lambda}\, r$

$$\frac{df}{dr} = \frac{df}{dz} \cdot \frac{dz}{dr} = \sqrt{\lambda}\ \frac{df}{dz}$$

$$\frac{d^2 f}{dr^2} = \lambda \frac{d^2 f}{dz^2}$$

$$r^2 \cdot \lambda \frac{d^2 f}{dz^2} + r \cdot \sqrt{\lambda} \frac{df}{dz} - n^2 f + \lambda r^2 f = 0$$

$$(**)\quad \underbrace{z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} - n^2 f + z^2 f = 0}_{\text{has no } \lambda.}$$

(7) Call (**) Bessel equation.

(**) has two solutions

$J_n(z)$ Bessel function of 1st kind

$Y_n(z)$ Bessel function of 2nd kind.

Behaviour of Bessel functions as $z \to +\infty$

$$z^2 f'' + z f' + \underbrace{(-n^2 + z^2)}_{z \text{ large } \sim z^2} f = 0$$

$$\underline{z^2} f'' + z f' + \underline{z^2} f = 0$$

$$z^2 \gg z.$$

compare with $z^2 f'' + z^2 f = 0$

Solution to $f'' + f = 0$

$$f = c_1 \sin z + c_2 \cos z.$$

J_n, Y_n are like $\sin, \cos$ having infinitely many zeros as $z \to +\infty$.

$z \to 0$, equation $(**)$ is

$$z^2 f'' + z f' + (n^2 + \underset{\substack{\uparrow \\ \text{small compare to } n^2.}}{z^2}) f = 0$$

compare with

$$z^2 f'' + z f' - n^2 f = 0$$

equidimensional ODE

$f = z^p$,

$p(p-1) + p - n^2 = 0 \Rightarrow p = \pm n$.

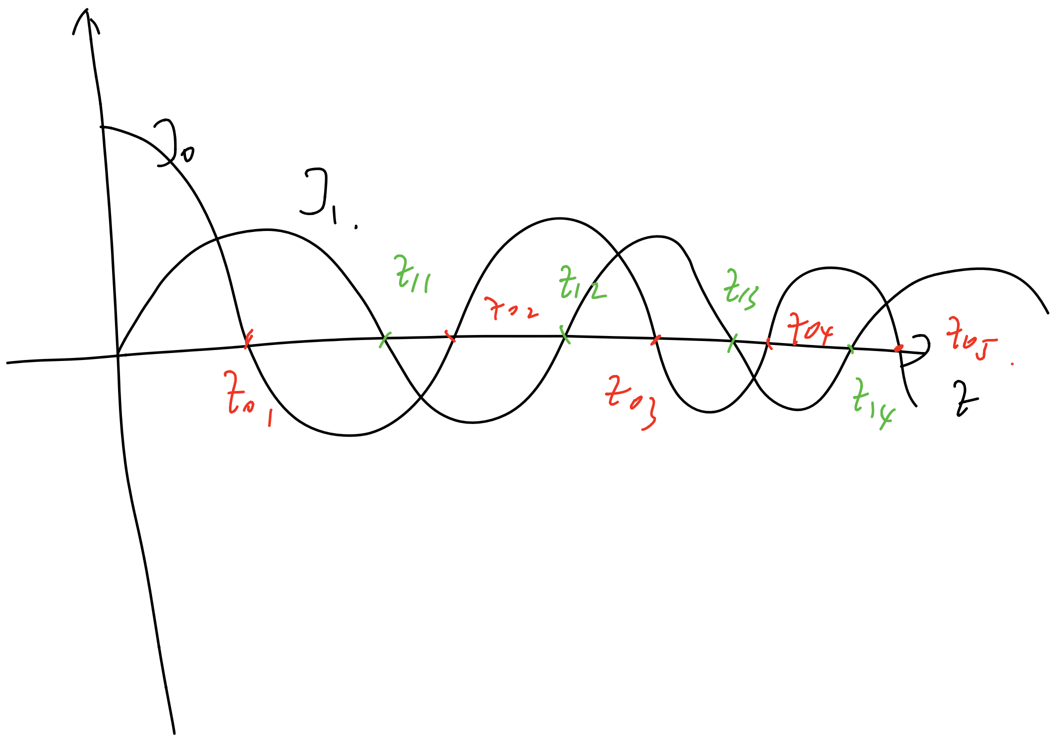
$f(z) = z^n$ or z^{-n}

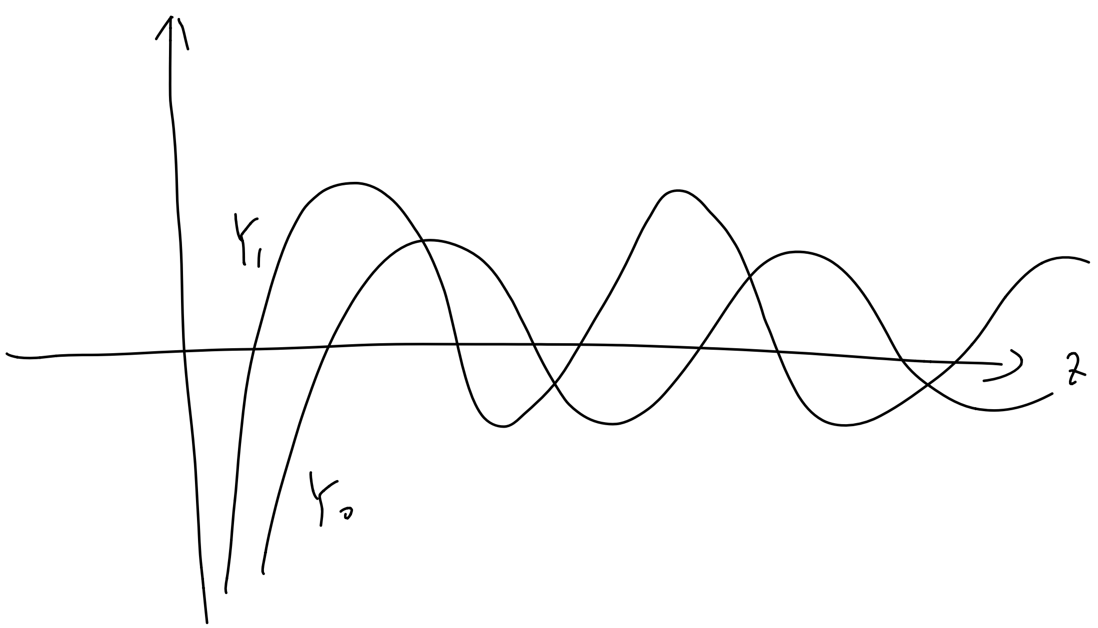
$n = 0$, $f = \log z$ or 1.

As $z \to 0$, $J_n(z) \sim \begin{cases} z^n & n \neq 0 \\ 1 & n = 0 \end{cases}$

$Y_n(z) \sim \begin{cases} z^{-n} & n \neq 0. \\ \log z & n = 0 \end{cases}$

Graphs





From $|f(0)| < +\infty$.

$$f(z) = J_n(z) = J_n(\sqrt{\lambda}\, r)$$

$r = R$, $f(R) = 0$, this implies

$\sqrt{\lambda}\, R = z_{nm}$, one of the zeros

so $\lambda = \left(\frac{z_{nm}}{R}\right)^2$.

Summary: $f(r) = J_n\left(\frac{z_{nm}}{R} r\right)$

$n = 0, 1, \ldots$

$m = 1, 2, \ldots$

$$\phi_{nm}(r, \theta) = C_1 J_n\left(\frac{z_{nm}}{R} r\right) \cos n\theta + C_2 J_n\left(\frac{z_{nm}}{R} r\right) \sin n\theta$$

Orthogonality, even though this not regular, but the proof for orthogonality does not depend on regularity.

$$\langle J_n\left(\frac{z_{nm}}{R} r\right), J_n\left(\frac{z_{nk}}{R} r\right) \rangle_r$$

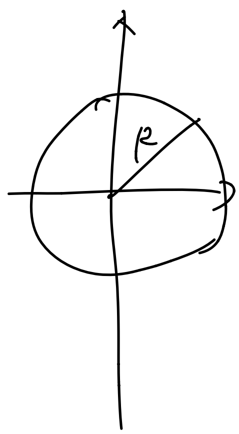
$$= \int_0^R J_n\left(\frac{z_{nm}}{R} r\right) \cdot J_n\left(\frac{z_{nk}}{R} r\right) \cdot r \, dr$$

$$= 0 \quad \text{if } m \neq k.$$

If $\sum_{m=1}^{+\infty} a_m J_n\left(\frac{z_{nm}}{R} r\right) = f(r)$

then $$a_m = \frac{\int_0^R f(r) \cdot J_n\left(\frac{z_{nm}}{R} r\right) r \, dr}{\int_0^R \left(J_n\left(\frac{z_{nm}}{R} r\right)\right)^2 r \, dr}$$

Example: Wave equation on 2D disc



$u(r, \theta, t)$.

$u_{tt} = c^2 \Delta u$

$u(R, \theta, t) = 0$ BC

$\begin{cases} u(r, \theta, 0) = f(r, \theta) \\ u_t(r, \theta, 0) = g(r, \theta) \end{cases}$ ICs

seperation of variables

$u(r, \theta, t) = \phi(r, \theta) \cdot G(t)$

$$\frac{\Delta \phi}{\phi} = -\lambda = \frac{G''(t)}{c^2 G(t)}.$$

$G(t) = C_1 \sin(\sqrt{\lambda}\, c t) + C_2 \cos(\sqrt{\lambda}\, c t)$

$$u(r,\theta,t) = \sum_{n=0}^{+\infty} \sum_{m=1}^{+\infty} A_{nm} \cdot J_n\left(\frac{z_{nm}}{R} r\right) \cdot \cos n\theta \cdot \cos\left(\frac{z_{nm}}{R} c t\right)$$

$$+ \sum_{n=0}^{+\infty} \sum_{m=1}^{+\infty} B_{nm} J_n\left(\frac{z_{nm}}{R} r\right) \cdot \cos n\theta \cdot \sin\left(\frac{z_{nm}}{R} c t\right)$$

$$+ \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} C_{nm} J_n\left(\frac{z_{nm}}{R} r\right) \sin n\theta \cdot \cos\left(\frac{z_{nm}}{R} c t\right)$$

$$+ \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} D_{nm} J_n\left(\frac{z_{nm}}{R} r\right) \sin n\theta \sin\left(\frac{z_{nm}}{R} c t\right)$$

(IC_1) $\quad u(r,\theta,0) = f(r,\theta)$

$$\Rightarrow A_{nm} = \frac{\int_{-\pi}^{\pi}\int_0^R f(r,\theta) \cdot J_n\left(\frac{z_{nm}}{R} r\right) \cdot \cos n\theta \, r \, dr \, d\theta}{\int_{-\pi}^{\pi}\int_0^R \left(J_n\left(\frac{z_{nm}}{R} r\right)\right)^2 (\cos n\theta)^2 \, r \, dr \, d\theta}$$

$$= \left\{ \frac{\int_{-\pi}^{\pi}\int_0^R f(r,\theta) \cdot J_0\left(\frac{z_{0m}}{R} r\right) r \, dr \, d\theta}{2\pi \cdot \int_0^R \left(J_0\left(\frac{z_{0m}}{R} r\right)\right)^2 r \, dr} \right.$$

$$\left(\frac{\int_{-\pi}^{\pi} \int_0^{12} f(r,\theta) \cdot J_n\left(\frac{z_{nm}}{12} r\right) \cos n\theta \, r \, dr \, d\theta}{\pi \cdot \int_0^{12} \left(J_n\left(\frac{z_{nm}}{12} r\right)\right)^2 r \, dr} \right.$$

B_{nm}, C_{nm}, $D_{n,m}$ similar formulas.