Laplace equation on a cylinder

In a general 3D region $\Omega$

$$\begin{cases} \Delta u = 0 \\ u\big|_{\partial\Omega} = f(x,y,z) & \text{Dirichlet BC} \\ \text{or } \dfrac{\partial u}{\partial \vec{n}}\Big|_{\partial\Omega} = g(x,y,z) & \text{Neumann BC.} \end{cases}$$

uniqueness of solution in $\begin{cases} \text{Dirichlet BC.} \\ \text{Neumann.} \end{cases}$

pf: $u_1, u_2$ are two solutions

$v = u_1 - u_2$. $\quad \Delta v = 0$, $\quad v\big|_{\partial\Omega} = 0$.

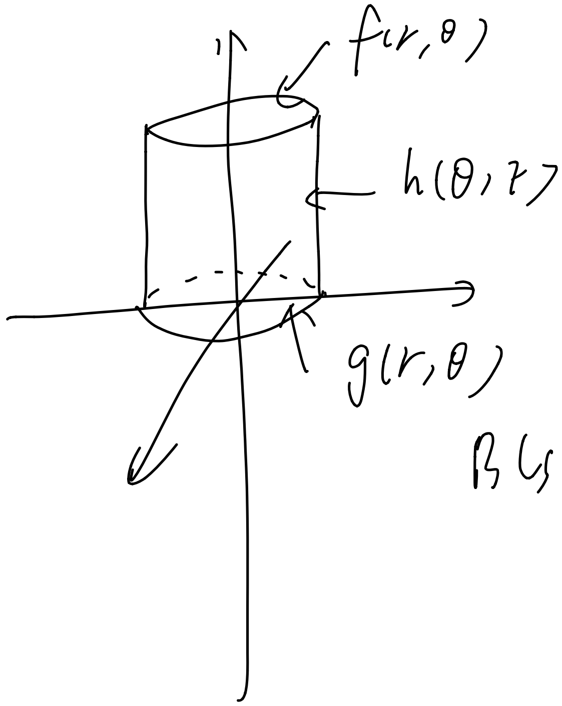
$$0 = \iiint_\Omega (\Delta v)\cdot v = \iiint_\Omega |\nabla v|^2 + \underbrace{\boxed{\iint_{\partial\Omega} \langle \nabla v, \vec{n} \rangle v}}_{= 0}$$

$\Rightarrow \quad |\nabla v| = 0 \quad \Rightarrow \quad v = \text{constant}.$

Dirichlet $\Rightarrow$ $v = 0$

Neumann $\Rightarrow$ $v = \text{constant}$.

Solve $\Delta u = 0$ on a cylinder $0 \le r \le R$
$0 \le z \le H$



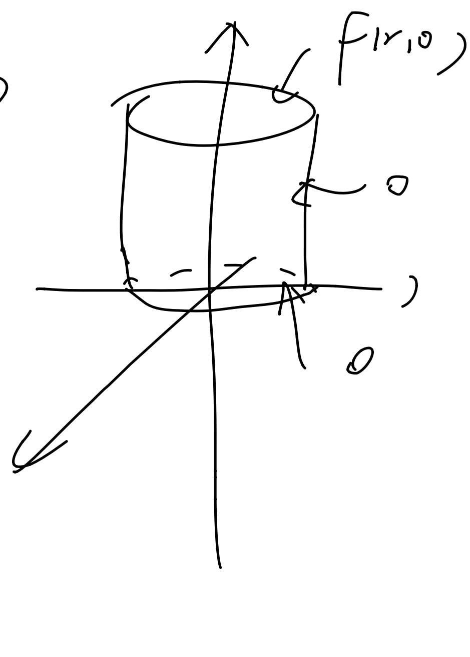

$\Delta u = 0$

$u(r,\theta,z)$

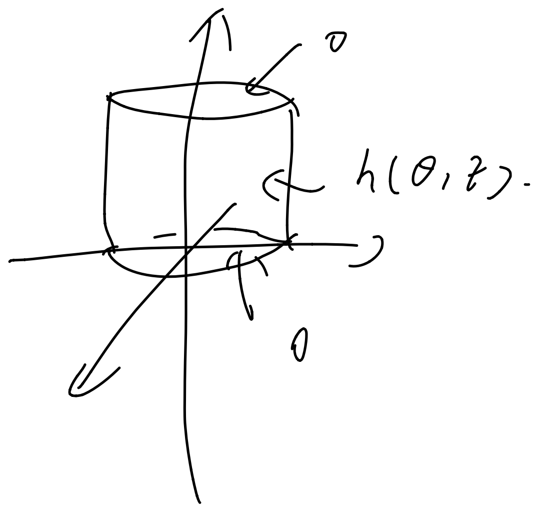
BCs
$$\begin{cases} u(r,\theta,H) = f(r,\theta) \\ u(r,\theta,0) = g(r,\theta) \\ u(R,\theta,z) = h(\theta,z) \end{cases}$$

Get more homogeneous conditions.

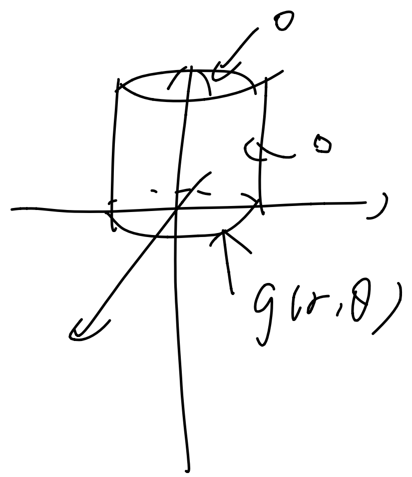
Case (1)



case (2)




case (3)




The same as case (1)

case (1). $(r,\theta)$ homogeneous variables.

$$u(r,\theta,z) = \phi(r,\theta)\cdot G(z).$$

$$\frac{\Delta\phi}{\phi} = -\lambda = -\frac{G''}{G}.$$

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi|_{\partial D} = 0 \end{cases}$$

$$\lambda_{nm} = \left(\frac{z_{nm}}{R}\right)^2 .$$

$$\phi_{nm} = J_n\left(\frac{z_{nm}}{R} r\right) \cos n\theta$$

$$J_n\left(\frac{z_{nm}}{R} r\right) \sin n\theta$$

$n = 0, 1, 2 \ldots$

$m = 1, 2, \ldots$

$$G''(z) = \lambda G(z)$$

$$G(z) = C_1 \sinh\sqrt{\lambda} z + C_2 \cosh\sqrt{\lambda} z$$

$$G(0) = 0 \Rightarrow C_2 = 0$$

$$G(z) = \sinh\sqrt{\lambda} z$$

$$\text{So } u(r,\theta,z) = \sum_{n=0}^{+\infty}\sum_{m=1}^{+\infty} A_{nm} J_n\left(\frac{z_{nm}}{R} r\right) \cos n\theta \, \sinh\left(\frac{z_{nm}}{R} z\right)$$

$$+ \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} B_{nm} J_n\left(\frac{z_{nm}}{R} r\right) \sin n\theta \, \sinh\left(\frac{z_{nm}}{R} z\right)$$

Match BC by orthogonality.

Case (2). $z$ is homogeneous variable. $u(r,\theta,z) = \phi(r,\theta)\cdot G(z)$

$$\frac{\Delta\phi}{\phi} = -\frac{G''}{G} = \lambda.$$

$$\begin{cases} G''(z) = -\lambda G(z). \\ G(0) = G(H) = 0 \end{cases}$$

$$\lambda = \left(\frac{n\pi}{H}\right)^2 \quad G(z) = \sin\frac{n\pi}{H}z.$$

$$\Delta\phi = \left(\frac{n\pi}{H}\right)^2\phi.$$

$$\frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta} = \left(\frac{n\pi}{H}\right)^2\phi.$$

$\theta$ is also homogeneous variable.

$$\phi(r,\theta) = f(r)\cdot g(\theta)$$

Then $\frac{1}{r}(rf')'g + \frac{1}{r^2} g'' f = (\frac{n\pi}{l-l})^2 fg$.

$$\frac{r(rf')'}{f} + \underbrace{\frac{g''}{g}}_{\text{constant } \mu} = (\frac{n\pi}{l-l})^2 r^2$$

$$\begin{cases} g''(\theta) = -\mu g \\ g(-\pi) = g(\pi) \\ g'(-\pi) = g'(\pi) \end{cases}$$

so $\mu = m^2$, $g(\theta) = c_1 \cos m\theta + c_2 \sin m\theta$

(*) $r^2 f'' + r f' - m^2 f - (\frac{n\pi}{l-l})^2 r^2 f = 0$

Different from Bessel because this term is negative.

Try to study (*) by using change of variable and new names for "standard" equation.

$$w = \left(\frac{n\pi}{L}\right) r.$$

Then (**) $w^2 f'' + w f' - m^2 f - w^2 f = 0$.

Call (**) modified Bessel equation.

Solutions are

$I_m(w)$ Modified Bessel function of 1st kind

$K_m(w)$ Modified Bessel function of 2nd kind.

Asymptotic behaviour.

$w \to +\infty$. Compare with

$$w^2 f'' + \underbrace{w f'}_{\text{small}} - w^2 f = 0$$

$$w^2 (f'' - f) = f$$

$$f = e^{w} \text{ or } e^{-w}$$

(No zeros)

$w \to 0$, compare with.

$$w^2 f'' + w f' - m^2 f = 0.$$

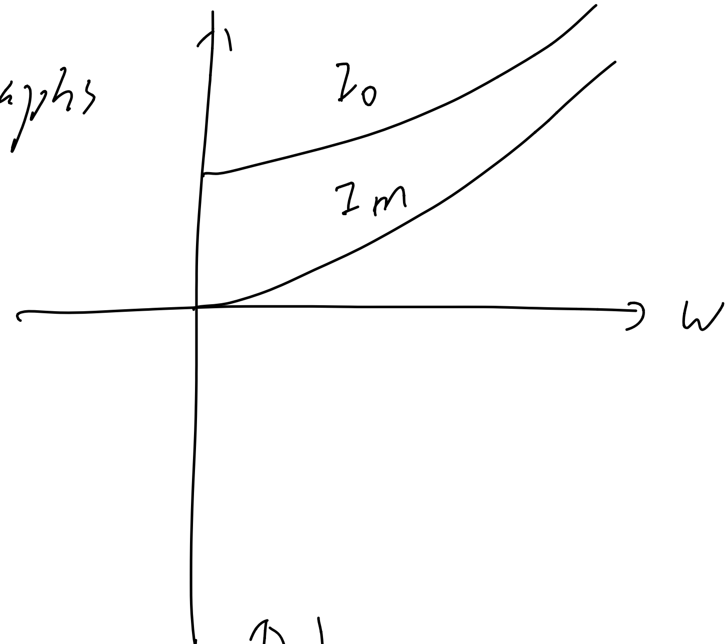
solve this equidimensional ODE.

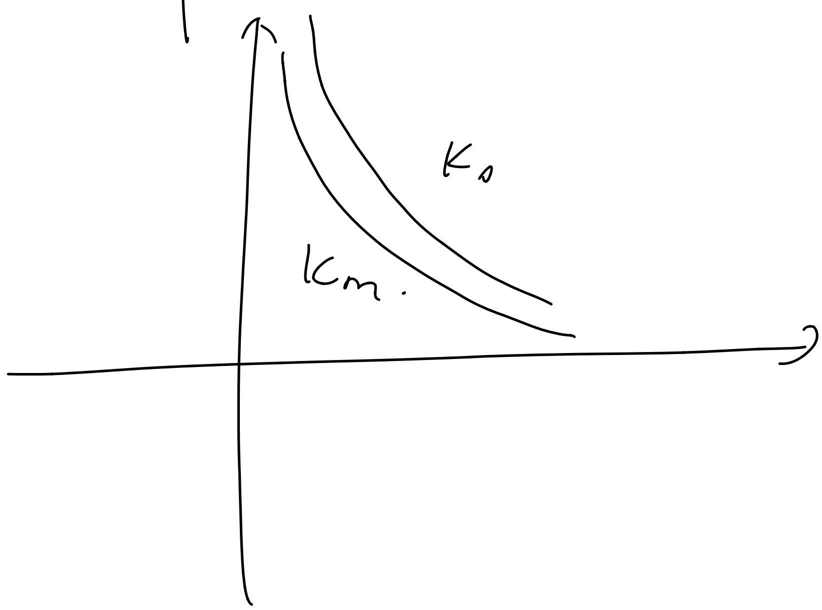
$$f = \begin{cases} w^m \text{ or } w^{-m} & m \neq 0 \\ \log w \text{ or } 1 & m = 0. \end{cases}$$

$$I_m(w) \sim \begin{cases} w^m & m \neq 0 \\ 1 & m = 0 \end{cases}$$

$$K_m(w) \sim \begin{cases} w^{-m} & m \neq 0 \\ \log w & m = 0 \end{cases}$$

Graphs




$$f(r) = C_1 I_m\left(\frac{n\pi}{l-l} r\right) + C_2 K_m\left(\frac{n\pi}{l-l} r\right)$$

since $|f(0)| < +\infty$, $C_2 = 0$

So $$u(r,\theta,z) = \sum_{n=1}^{+\infty} \sum_{m=0}^{+\infty} A_{nm} \sin \frac{n\pi}{H} z \cdot \cos m\theta \cdot I_m\left(\frac{n\pi}{H} r\right)$$

$$+ \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} A_{nm} \sin \frac{n\pi}{H} z \sin m\theta \cdot I_m\left(\frac{n\pi}{H} r\right)$$

Match the BC $u(R,\theta,z) = h(\theta,z)$