Lec 2.

Multivariable calculus.

$f(x,y,z)$

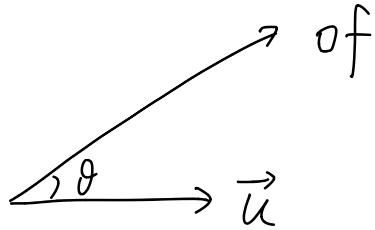
Gradient $\nabla f = \langle f_x, f_y, f_z \rangle$.

Directional derivative in direction $\vec{u} = \langle u_1, u_2, u_3 \rangle$

$(|\vec{u}| = 1)$

$$D_{\vec{u}} f = \langle \nabla f, \vec{u} \rangle = f_x \cdot u_1 + f_y \cdot u_2 + f_z \cdot u_3.$$

$$= |\nabla f| \cdot |\vec{u}| \cdot \cos\theta.$$



$\vec{u} = \dfrac{\nabla f}{|\nabla f|}$ is the direction in which f increases fastest.

(because $\theta = 0$ and $\cos\theta = 1$ in this case)

$$\vec{F} = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z) \rangle.$$

is a vector field,

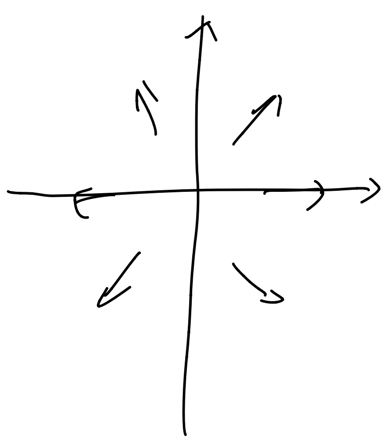
$$\operatorname{div} \vec{F} = P_x + Q_y + R_z = \text{"divergence of } F\text{."}$$

$\operatorname{div} \vec{F} > 0$, source of the flow $\vec{F}$.

$\operatorname{div} \vec{F} < 0$, sink of the flow $\vec{F}$.

Ex: $f(x,y) = x^2 + y^2$.

$\nabla f = \langle 2x, 2y \rangle$.

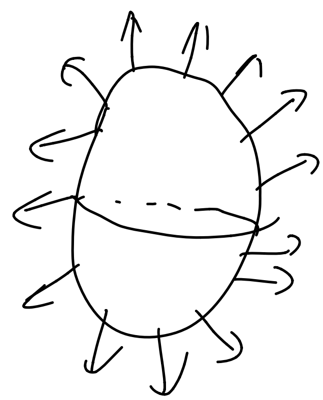


$\Delta f = \operatorname{div}(\nabla f) = 2 + 2 = 4$.

Divergence theorem: $\Omega \subset \mathbb{R}^3$ bounded region.

$\partial\Omega$ boundary. $\vec{F}$ vector field in $\mathbb{R}^3$.

$$\iiint_{\Omega} \operatorname{div} \vec{F} = \iint_{\partial\Omega} \langle \vec{F}, \vec{n} \rangle .$$



$$\int_{\Omega} \operatorname{div} \vec{F} = \int_{\partial\Omega} \langle \vec{F} \cdot \vec{n} \rangle$$

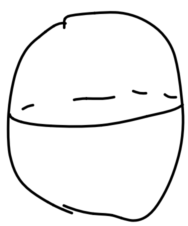
(Notation: usually abbreviate triple/double integrals

$$\iiint_{\Omega} \longrightarrow \int_{\Omega} \quad \text{for convenience}$$

$$\iint_{\partial\Omega} \longrightarrow \int_{\partial\Omega}$$

)

Ex: $\vec{F} = \langle x, y, z \rangle$. $\operatorname{div} \vec{F} = 1+1+1 = 3$



Ω unit ball

$$\int_{\partial\Omega} \langle \vec{F}, \vec{n} \rangle = \int_{\Omega} 3$$

$$= 3 \cdot \text{volume}$$

$$= 3 \cdot \frac{4\pi}{3} = 4\pi$$

Integration by parts:

$$\int_a^b f \cdot g' = f \cdot g \Big|_a^b - \int_a^b f' g .$$

Pf: $(fg)' = f'g + g'f$.

$$\int_a^b (fg)' = \int_a^b f'g + \int_a^b f'g$$

$$\parallel$$

$$fg \Big|_a^b \quad \text{(rearrange)}.$$

higher dimensional version

$$\int_{\Omega} f \Delta g = \int_{\partial\Omega} f \langle \nabla g, \vec{n} \rangle - \int_{\Omega} \langle \nabla f, \nabla g \rangle.$$

Pf. "Product rule" (Homework).

$$\operatorname{div}(f \nabla g) = f \Delta g + \langle \nabla f, \nabla g \rangle.$$

$$\left(\nabla \cdot (f \nabla g) = f \nabla^2 g + \langle \nabla f, \nabla g \rangle \right).$$

Div. thm.

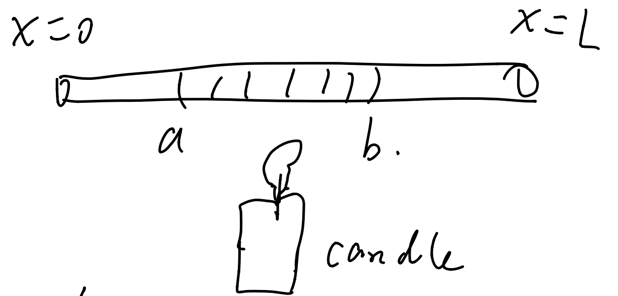
$$\int_{\partial\Omega} \langle f \nabla g, \vec{n} \rangle = \int_{\Omega} \operatorname{div}(f \nabla g)$$

$$\int_{\partial\Omega} f \langle \nabla g, \vec{n} \rangle = \int_{\Omega} f \Delta g + \int_{\Omega} \langle \nabla f, \nabla g \rangle.$$

Heat equation:
- Derivation of heat equation.
- Boundary conditions

$u(x,t)$.

temp at point x and time t.



$e(x,t)$ = thermal energy density

$\rho(x)$ = mass density

$c(x)$ = specific heat

(the heat energy that must be supplied to a unit mass of substance to raise its temp one unit)

$Q(x,t)$ = heat energy density generated in rod per unit time

$\phi(x,t)$ = heat flux

(thermal energy per unit time flowing from left to right)

Total change of heat energy

= Total heat energy flowing across boundary per time

+ energy generated inside per unit time

$$\frac{d}{dt}\int_a^b e(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b Q\,dx$$

$$\int_a^b e_t \, dx = -\int_a^b \phi_x \, dx + \int_a^b Q \, dx$$

So $\boxed{e_t = -\phi_x + Q}$ differential equation.

$$e(x,t) = C \cdot \rho \, u(x,t)$$

$\boxed{\phi(x,t) = \underbrace{-K_0}_{\text{Thermal conductivity.}} u_x}$ → Fourier's law.

Heat flows from hot to cold.

So $C(x)\rho(x) \cdot u_t = (K_0 u_x)_x + Q$.

If C, ρ, K_0 are constants,

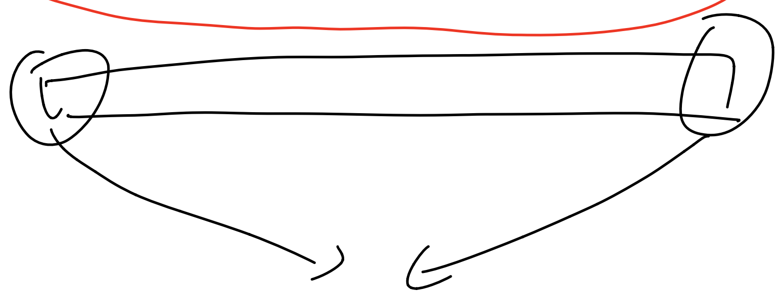
$Q = 0$

then $u_t = k u_{xx}$ where $k = \dfrac{K_0}{\rho \cdot C}$

Boundary conditions:

(1) Prescribed temperature.

"Dirichlet conditions"



ice water.

$$u(0,t)=0, \quad u(L,t)=0.$$

or some other functions of t.

(2) Insulated boundary "Neumann conditions".

$$u_x(0,t)=0, \quad u_x(L,t)=0.$$

$$\left(\phi(0,t)=\phi(L,t)=0\right).$$