Spherical problems

3D Laplace eigenvalue problem.

Ω 3D region.

$w(x,y,z)$ $\quad \Delta w = w_{xx} + w_{yy} + w_{zz}$

$$\begin{cases} \Delta w + \lambda w = 0 \\ w\big|_{\partial\Omega} = 0 \end{cases}$$

When $\Omega = [0, L_1] \times [0, L_2] \times [0, L_3]$

Then $\lambda_{n_1,n_2,n_3} = \underbrace{\left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2 + \left(\frac{n_3\pi}{L_3}\right)^2}_{\text{sum}}.$

$n_1, n_2, n_3 = 1, 2, \ldots$

$$w_{n_1,n_2,n_3} = \underbrace{\left(\sin\frac{n_1\pi}{L_1}x\right)\left(\sin\frac{n_2\pi}{L_2}y\right)\left(\sin\frac{n_3\pi}{L_3}z\right)}_{\text{product.}}$$

If $\Omega = \{(x,y) \mid x^2+y^2 \le R^2\} \times [0, H)$.

$$\lambda_{n,m,k} = \underbrace{\left(\frac{z_{nm}}{R}\right)^2 + \left(\frac{k\pi}{H}\right)^2}.$$

$$\phi_{n,m,k} = \underline{J_n\left(\frac{z_{nm}}{R} r\right) \cdot \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \cdot \sin \frac{k\pi z}{H}}.$$

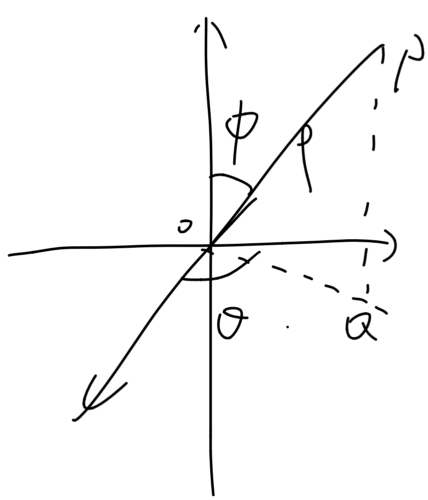
$n = 0, 1, 2, \cdots$, $m = 1, 2, \cdots$

$k = 1, 2, 3, \cdots$

Remark: If Ω is a product, we always have this proposition

λ = sum. ϕ = product

$\Omega: x^2+y^2+z^2 = R^2$.



$\rho = |OP|$

ϕ = angle between z-axis and $\vec{OP}$

θ = angle of $\vec{OQ}$ in polar coordinate.

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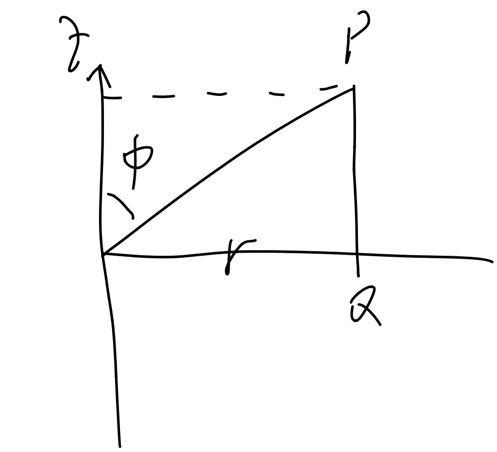

$x = r\cos\theta$

$y = r\sin\theta$

$r = \rho \cdot \sin\phi.$

$z = \rho\cos\phi.$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta.$

$\rho \geq 0, \ 0 \leq \phi \leq \pi, \ -\pi \leq \theta \leq \pi.$

$$\Delta w = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial w}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial w}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 w}{\partial\theta^2} = -\lambda w.$$

$$w(\rho, \theta, \phi) = f(\rho)\,\underline{q(\theta)}\,g(\phi)$$

$$\frac{q''(\theta)}{q(\theta)} = \text{const}, \quad q(-\pi) = q(\pi), \quad q'(-\pi) = q'(\pi)$$

$\downarrow$

so this is m^2.

So: $\frac{1}{f} \frac{d}{d\rho}(\rho^2 f') + \lambda\rho^2$

$$= -\frac{1}{g\sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right)$$

$$+ \frac{m^2}{\sin^2\phi} = \mu.$$

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + (\lambda\rho^2 - \mu)f = 0$$

$$0 \le \rho \le R.$$

$$(*) \quad \frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \left(\mu\sin\phi - \frac{m^2}{\sin\phi}\right)g = 0.$$

$$0 \le \phi \le \pi.$$

Focus on $(*)$. SL problem,

but not regular.

Change of variable

$x = \cos\phi$, $\quad \frac{dx}{d\phi} = -\sin\phi$

so: $\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \left(\mu - \frac{m^2}{1-x^2}\right)y = 0$

When $x = \pm 1$, $1-x^2 = 0$, $\frac{m^2}{1-x^2} \to +\infty$

$|y(1)|, |y(-1)| < +\infty$.

When $x = 1$, $x^2 - 1 = (x-1)(x+1)$

$\sim 2(x-1)$.

$\mu - \frac{m^2}{1-x^2} \sim -\frac{m^2}{2(1-x)}$

$(1-x^2) \sim 2(1-x)$.

compare with

$$-2\frac{d}{dx}\left((x-1)\frac{dy}{dx}\right)+\frac{m^2}{2(x-1)}=0.$$

equidimensional with respect to

$t=x-1.$

$$-2\frac{d}{dt}\left(t\frac{dy}{dt}\right)+\frac{m^2}{2t}=0$$

$$y=t^p \Rightarrow -2p^2t^{p-1}+\frac{m^2}{2}t^{p-1}=0.$$

$$-2p^2+\frac{m^2}{2}=0 \Rightarrow p=\pm\frac{m}{2}.$$

$g \sim (x-1)^{\frac{m}{2}}$ when $x \to 1$.

or $(x-1)^{-\frac{m}{2}} \to \infty$ when $x \to 1$.

So one solution is finite.

the other solution is infinite.