Last time.

$w(\rho, \theta, \phi)$.

$$\begin{cases} \Delta w + \lambda w = 0 \\ w(R, \theta, \phi) = 0 \end{cases}$$

$$w(\rho, \theta, \phi) = f(\rho)\, q(\theta)\, g(\phi)$$

$$\begin{matrix} q(\pi) = q(-\pi) \\ q'(\pi) = q'(-\pi) \end{matrix} \Rightarrow \frac{q''(\theta)}{q(\theta)} = -m^2$$

$$q(\theta) = \begin{cases} \cos m\theta \\ \sin m\theta \end{cases}.$$

$$(*)\ \frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + (\lambda \rho^2 - \mu) f = 0.$$

$$(**)\ \frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \left(\mu \sin\phi - \frac{m^2}{\sin\phi}\right) g = 0.$$

$x = \cos\phi \Rightarrow$

$$\frac{d}{dx}\left[(1-x^2)\frac{dg}{dx}\right] + \left(\mu - \frac{m^2}{1-x^2}\right) g = 0$$

$x = 1$, $\quad 1-x^2 \sim 2(1-x)$

compare with. $\quad \mu - \dfrac{m^2}{1-x^2} \sim \mu - \dfrac{m^2}{2(1-x)}$

$$\frac{d}{dx}\left[2(1-x)\frac{dg}{dx}\right] - \frac{m^2}{2(1-x)} g = 0.$$

$$g(x) \sim \begin{cases} (1-x)^{\frac{m}{2}} \\ (1-x)^{-\frac{m}{2}}. \end{cases}$$

$|g(1)| < +\infty$ cut down dimension by one

$|g(-1)| < +\infty$ cut down dimension by one

determines some special μ.

Answer: $\mu = \underline{n(n+1)}$. $\quad n = m, m+1, \ldots$

Associated Legendre function.

Frobenius method: $\boxed{m=0}$.

$$g(x)=\sum_{n=0}^{+\infty} a_n x^n. \qquad g'(x)=\sum_{n=0}^{+\infty} n a_n x^{n-1}$$

$$g''(x)=\sum_{n=0}^{+\infty} n(n-1) a_n x^{n-2}$$

$$(1-x^2)\sum_{n=0}^{+\infty} n(n-1)\, a_n x^{n-2} - 2x\sum_{n=0}^{+\infty} n a_n x^{n-1}$$

$$+\mu \sum_{n=0}^{+\infty} a_n x^n = 0.$$

$$\sum_{n=0}^{+\infty} (n+2)(n+1)\, a_{n+2}\, x^n - \sum_{n=0}^{+\infty} n(n-1)\, a_n x^n$$

$$-\sum_{n=0}^{+\infty} 2n a_n x^n + \mu \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$(n+2)(n+1)\, a_{n+2} - \left(n(n-1)+2n-\mu\right) a_n = 0$$

$$a_{n+2} = \frac{n(n+1)-\mu}{(n+2)(n+1)} a_n$$

$$\sim \frac{n}{n+2}.$$

even solution:

More accurate estimate.

$$\frac{n(n+1)-\mu}{(n+2)(n+1)} \to \frac{n-1}{n+1}$$

for n large enough.

$$a_2 = \frac{-\mu}{2} a_0$$

$$a_4 = \frac{-\mu + 3\cdot 2}{4\cdot 3} a_2$$

$\vdots$

$$a_{2n+2} = (\quad)\cdot \boxed{\frac{-\mu + 2n(2n+1)}{(2n+2)(2n+1)}} a_0.$$

approximately $\frac{2n}{2n+2} = \frac{n}{n+1}$

even $g_1(x) = a_0 \cdot \left(\sum_{n=0}^{+\infty} \square x^{2n}\right)$

odd $g_2(x) = a_1 \left(\sum_{n=1}^{+\infty} \square x^{2n-1}\right)$

$\square \sim \frac{1}{n}$.

$g_1(x)$, $g_2(x)$ both diverges $\to \pm\infty$ at $x = \pm 1$

because $\sum_{n=1}^{+\infty} \frac{1}{n} \to +\infty$.

Only way to get a solution which is not going to infinity at both $x = \pm 1$, is to have

$$\mu = n(n+1)$$

because $a_{n+2} = \dfrac{-\mu + n(n+1)}{(n+1)(n+2)} a_n$,

in which case $a_{n+2} = 0$, $a_{n+4} = 0$ ...

solution is a polynomial

For $m \neq 0$, similar argument still holds, $\mu = n(n+1)$.

$n = 0$, $P_0(x) = 1$

$n = 1$, $P_1(x) = x = \cos\phi$.

$n=2$, $P_2(x) = \frac{1}{2}(3x^2-1)$

Rodrigues' formula.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n .$$

For $m \neq 0$,

$$y = (1-x^2)^{-m/2} g .$$

$$g = (1-x^2)^{m/2} y .$$

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right) - \frac{m^2}{1-x^2} g + \mu g = 0$$

(Straightforward but tedious calculation) see the code on course website

$$\Rightarrow \left(\frac{1}{1-x^2}\right)^{m/2} \left((1-x^2) y'' - 2(1+m) xy' + (\mu - m(m+1))y \right) = 0 .$$

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

Same idea gives.

$$a_{n+2} = \frac{2(m+1)\cdot n + n(n-1) - (\mu - m(m+1))}{(n+2)(n+1)} a_n$$

$$= \frac{n(n+2m+1) - (\mu - m(m+1))}{(n+2)(n+1)} a_n$$

$$\approx \frac{(n-1)(n+2m)}{(n+2)(n+1)}$$

for n large.

$$\frac{(2n-1)(2n+2m)}{(2n+2)(2n+1)} \cdot \frac{(2n-3)(2n-2+2m)}{\cancel{(2n)}(2n-1)} \cdot \frac{(2n-5)(2n-4+2m)}{(2n-2)(2n-3)} \cdots$$

$$= \frac{(2n+2m)\cdot(2n-2+2m)\cdots}{(2n+2)}$$

$$= \frac{(m+n)(m+n-1)\cdots m}{(n+1)!}$$

compare with power series

$$\left(\frac{1}{1-x^2}\right)^m = \sum \frac{m(m+1)\cdots(m+n-1)}{n!} x^{2n}.$$

So the solution $y(x)$ will blow up like $\left(\frac{1}{1-x^2}\right)^m$ at $x=\pm 1$

Just as before, special value of μ will give zero in the recursive coefficient.

$$\begin{aligned}
\mu &= m(m+1) + n(n+2m+1) \\
&= m(m+1) + n(m+1) + n(n+m) \\
&= (m+1)(n+m) + n(n+m) = (n+m+1)(n+m)
\end{aligned}$$

In other words, $\mu = n(n+1)$, $n = m, m+1, \ldots$

Solutions are $y(x) = \frac{d^m}{dx^m} P_n(x)$

So $P_n^m(x) = (1-x^2)^{m/2} \cdot \frac{d^m}{dx^m} P_n(x)$.

is the solution to

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \left(\mu - \frac{m^2}{1-x^2}\right) y = 0.$$

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \left(\lambda\rho^2 - n(n+1)\right) f = 0$$

change of variable

$z = \sqrt{\lambda}\,\rho \Rightarrow$

$$z^2 f''(z) + 2z f'(z) + (\lambda z^2 - n(n+1)) f = 0$$

(different from Bessel equation)

$$f(z) = z^{-1/2} F(z).$$

Then $z^2 F''(z) + z F'(z) + (\lambda z^2 - (n+\frac{1}{2}))F = 0.$

$$F(z) = J_{n+\frac{1}{2}}(z)$$

$$f(\rho) = \rho^{-\frac{1}{2}} \cdot J_{n+\frac{1}{2}}(\sqrt{\lambda}\,\rho).$$

Determine λ by zeros of

$J_{n+\frac{1}{2}}$.

Spherical harmonics

Eigen functions.

$$\rho^{-\frac{1}{2}} \cdot J_{n+\frac{1}{2}}(\sqrt{\lambda}\,\rho) \left\{ \begin{matrix} \cos m\theta \\ \sin m\theta \end{matrix} \right\} \cdot P_n^m(\cos\phi)$$

$m = 0, 1, \ldots$

$n = m, m+1, \ldots$