Δw in spherical coordinates.

$$\Delta w = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial}{\partial \rho} w\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial}{\partial \phi}\right)$$

$$+ \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 w}{\partial \theta^2}$$

$$= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial}{\partial \rho} w\right) + \frac{1}{\rho^2} \cdot \Delta_{(\phi,\theta)}.$$

$\Delta_{(\phi,\theta)}$ is the Laplacian on the unit sphere S^2.

Eigenvalue problem:

$$\Delta_{(\phi,\theta)} f + \mu f = 0$$

$$\mu_n = n(n+1). \quad n = 0, 1, 2, \ldots$$

Eigenfunctions for one μ_n are

$$Y_n^m = P_n^m(\cos\phi) \cdot \begin{cases} \cos m\theta \\ \sin m\theta \end{cases}$$

$m = 0, 1, 2, \cdots n.$

$$\int_0^{\pi}\int_0^{2\pi} Y_n^m Y_{n'}^{m'} (\sin\phi)\, d\theta\, d\phi$$

$$= \begin{cases} 0 & \text{if } Y_n^m \neq Y_{n'}^{m'} \\ \left(n+\frac{1}{2}\right)^{-1} \dfrac{(n+m)!}{(n-m)!} & \text{if } Y_n^m = Y_{n'}^{m'}. \end{cases}$$

In ρ-direction:

$$\frac{d}{d\rho}\left(\rho^2 f'\right) + \left(\lambda\rho^2 - n(n+1)\right)f = 0.$$

$$f = j_{n+\frac{1}{2}}(\sqrt{\lambda}\rho) = \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho)$$

Laplace equation in a Spherical Cavity:

$$\begin{cases} \Delta u = 0 \\ u(a, \theta, \phi) = f(\theta, \phi) \end{cases}$$

$$\Delta = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2} \Delta_{(\theta,\phi)} u = 0$$

$$u = F(\theta, \phi) \cdot G(\rho).$$

$$\Delta u = 0 \Rightarrow \frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial G}{\partial \rho}\right) \cdot F$$

$$+ \left(\Delta_{(\theta,\phi)} F\right) \cdot G(\rho) = 0$$

$$\Rightarrow \frac{\frac{d}{d\rho}\left(\rho^2 \frac{dG}{d\rho}\right)}{G} + \underbrace{\frac{\Delta_{(\theta,\phi)} F}{F}}_{-\mu_n = -n(n+1)} = 0.$$

So $$\frac{d}{d\rho}\left(\rho^2 \frac{dG}{d\rho}\right) - n(n+1)G = 0$$

$$G(\rho) = \rho^p,$$

$$p(p+1) = n(n+1)$$

$$\Rightarrow p = n \text{ or } (-n-1).$$

$$|G(0)| < +\infty \Rightarrow G(\rho) = \rho^n.$$

$$\underline{\rho^n \cdot Y_n^m.}$$

$$u(\rho,\theta,\phi) = \sum_{m=0}^{+\infty}\sum_{n=m}^{+\infty} A_{m,n}\,\rho^n \cdot P_n^m(\cos\phi)\cdot\cos m\theta$$

$$+ \sum_{m=1}^{+\infty}\sum_{n=m}^{+\infty} B_{m,n}\,\rho^n P_n^m(\cos\phi)\sin m\theta.$$

Nonhomogeneous problems:

Key for homogeneous BCs and PDE.

$u_1, u_2, \ldots$ satisfy the BCs and PDE.

Then so does $\sum_{n=1}^{+\infty} a_n u_n$

Heat eqn: $u(x,t)$

Ex: $u_t = k u_{xx}$ (PDE)

$u(0,t) = A$

$u(L,t) = B.$ (BCs)

$u(x,0) = f(x).$ (IC)

① Find special solution. u_E (equilibrium solution)

u_E (BCs) + (PDE)

② $w = u - u_E$ solves PDE + (BCs = 0) + Modified IC

$$u_E = A + \frac{B-A}{L} x.$$

$$\begin{cases} w_t = k w_{xx} \\ w(0,t) = 0 \\ w(L,t) = 0 \\ w(x,0) = f(x) - \left(A + \frac{B-A}{L} x\right) \end{cases}$$

Solve w. $\Rightarrow$ $u = w + u_E$.

Ex: Also works when we have heat source $Q(x)$

$$\begin{cases} u_t = u_{xx} + 2. & \text{PDE} \\ u(0,t) = A & \\ u(L,t) = B & \text{BCs} \\ u(x,0) = f(x) & \text{IC} \end{cases}$$

$u_E(x)$, $\underline{u'' + 2 = 0 \quad u(0) = A,\ u(L) = B.}$

$w = u - u_E$. (PDE, BCs)

What if $Q(x,t)$ depends on t.

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin 3x. \qquad x \in [0, \pi] \\ u(0,t) = 0 \\ u(\pi, t) = 1 \\ u(x,0) = f(x). \end{cases}$$

Method of eigen function expansion.

① First make BCs homogeneous.

$$u_0 = \frac{x}{\pi}.$$

$$w = u(x,t) - \frac{x}{\pi}.$$

$$w_t = w_{xx} + e^{-t}\sin x$$

$$\left.\begin{aligned} w(0,t) &= 0 \\ w(\pi,t) &= 0 \end{aligned}\right\} (BCs).$$

$$w(x,0) = f(x) - \frac{x}{\pi}.$$

②: $W(x,t) = \sum_{n=1}^{+\infty} A_n(t) \cdot \sin(nx)$.

$$W_t = \sum_{n=1}^{+\infty} A_n'(t) \sin nx$$

$$W_{xx} = -n^2 \sum_{n=1}^{+\infty} A_n(t) \sin(nx)$$

$$\begin{cases} A_n'(t) + n^2 A_n(t) = 0 & n \neq 3 \\ A_n'(t) + 3^2 A_n(t) = e^{-t} & n = 3 \end{cases}$$

$n \neq 3$ $\quad A_n(t) = e^{-n^2 t} \cdot C_n$.

$$A_n(0) = C_n \Rightarrow A_n(t) = e^{-n^2 t} \cdot A_n(0)$$

$n = 3$

$$e^{9t} \cdot A_3'(t) + 3^2 \cdot e^{9t} \cdot A_3(t)$$

$$= e^{8t}$$

$$(e^{9t} \cdot A_3(t))' = e^{8t}$$

$$e^{9t} A_3(t) = \frac{1}{8} e^{8t} + C$$

$$A_3(t) = \frac{1}{8} e^{-t} + C \cdot e^{-9t} .$$

$$A_3(0) = \frac{1}{8} + C .$$

$$C = A_3(0) - \frac{1}{8} .$$