Non-homogeneous

What if $Q(x,t)$ depends on t.

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin 3x. & x \in [0, \pi] \\ u(0,t) = 0 \\ u(\pi, t) = 1 \\ u(x,0) = f(x). \end{cases}$$

Method of eigen function expansion.

① First make BCs homogeneous.

$$u_0 = \frac{x}{\pi}.$$

$$w = u(x,t) - \frac{x}{\pi}.$$

$$w_t = w_{xx} + e^{-t}\sin x$$

$$\left.\begin{array}{l} w(0,t) = 0 \\ w(\pi,t) = 0 \end{array}\right\} \text{(BCs)}.$$

$$w(x,0) = f(x) - \frac{x}{\pi}.$$

②: $W(x,t) = \sum_{n=1}^{+\infty} A_n(t) \cdot \sin(nx)$.

$$W_t = \sum_{n=1}^{+\infty} A_n'(t) \sin nx$$

$$W_{xx} = -n^2 \sum_{n=1}^{+\infty} A_n(t) \sin(nx)$$

$$\begin{cases} A_n'(t) + n^2 A_n(t) = 0 & n \neq 3 \\ A_n'(t) + 3^2 A_n(t) = e^{-t} & n = 3 \end{cases}$$

$n \neq 3$ $\quad A_n(t) = e^{-n^2 t} \cdot C_n$.

$$A_n(0) = C_n \Rightarrow A_n(t) = e^{-n^2 t} \cdot A_n(0)$$

$n = 3$

$$e^{9t} \cdot A_3'(t) + 3^2 \cdot e^{9t} \cdot A_3(t)$$

$$= e^{8t}$$

$$(e^{9t} \cdot A_3(t))' = e^{8t}$$

$$e^{9t} A_3(t) = \frac{1}{8} e^{8t} + C$$

$$A_3(t) = \frac{1}{8} e^{-t} + C \cdot e^{-9t}.$$

$$A_3(0) = \frac{1}{8} + C.$$

$$C = A_3(0) - \frac{1}{8}.$$

$$w(x,t) = \left(\frac{1}{8} e^{-t} + \left(A_3(0) - \frac{1}{8}\right) e^{-9t} \right) \cdot \sin 3x$$

$$+ \sum_{\substack{n=1 \\ n \neq 3}}^{+\infty} e^{-n^2 t} \cdot \sin nx.$$

$$A_n(0) = \frac{2}{\pi} \int_0^{\pi} \left(f(x) - \frac{x}{\pi} \right) \cdot \sin nx \, dx.$$

Higher dim : (Forced vibrating membrane)

$$u_{tt} = c^2 \Delta u + Q(x,y,t)$$

$$\begin{cases} u\big|_{\partial\Omega} = 0 . \\ u(x,y,0) = f(x,y) . \quad u_t(x,y,0) = g(x,y) \end{cases}$$

$$\begin{cases} \Delta\phi + \lambda\phi = 0 \\ \phi\big|_{\partial\Omega} = 0 \end{cases}$$ $\phi(x,y)$ eigenfunction.

eigen values λ_n.

eigen functions $\phi_n(x,y)$.

$$Q(x,y,t) = \sum_{n=1}^{+\infty} q_n(t)\phi_n$$

$$q_n(t) = \frac{\iint_\Omega Q \cdot \phi_n \, dxdy}{\iint (\phi_n)^2 \, dxdy} .$$

$$u(x,y,t) = \sum a_n(t) \cdot \phi_n(x,y)$$

$$\sum_n a_n''(t)\,\phi_n = \sum \left(c^2(-\lambda_n)\, a_n(t) + q_n(t) \right) \phi_n .$$

$$\boxed{\left(a_n''(t) + c^2 \lambda_n a_n(t) \right) = q_n(t).}$$

Variation of coefficients.

$$\Rightarrow a_n(t) = c_1 \sin(c\sqrt{\lambda_n}\, t) + c_2 \cos(c\sqrt{\lambda_n}\, t)$$
$$+ \int_0^t q_n(z) \cdot \frac{\sin(c\sqrt{\lambda_n}(t-z))}{c\sqrt{\lambda_n}}\, dz .$$

In many cases, you can guess what the solutions should look like.

Periodic force:

$$Q(x,y,t) = Q(x,y) \cdot \cos wt.$$

$$Q(x,y) = \sum_{n=1}^{+\infty} \gamma_n \phi_n(x,y)$$

$$\gamma_n = \frac{\iint Q(x,y) \cdot \phi_n}{\iint (\phi_n)^2}$$

$$a_n''(t) + c^2 \lambda_n a_n(t) = \gamma_n \cos wt.$$

Guess $a_n(t) = B_n \cdot \cos wt.$

$$(-w^2 + c^2\lambda_n) B_n \cos wt = \gamma_n \cos wt.$$

$$B_n = \frac{\gamma_n}{c^2\lambda_n - w^2} \quad (\text{If } c^2\lambda_n \neq w^2)$$

$$a_n(t) = C_1 \cos\sqrt{\lambda_n}\, ct + C_2 \sin\sqrt{\lambda_n}\, ct + \frac{\gamma_n}{c^2\lambda_n - w^2} \cos wt.$$

If $c^2\lambda_n = \omega^2$, then

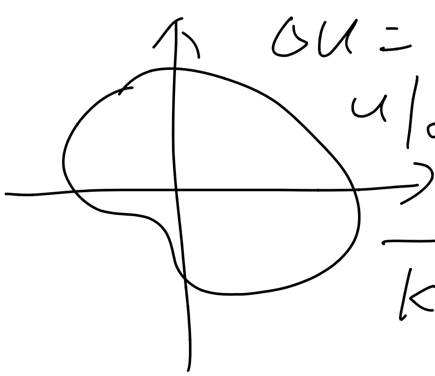
$$a_n(t) = \frac{f_n}{2\omega}\, t \sin(\omega t)$$

t goes to infinity.

(Resonance)

Poisson equation.

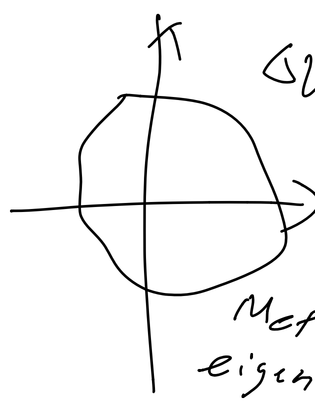
$$\begin{cases} \Delta u = Q(x,y) \\ u|_{\partial\Omega} = \alpha(x,y) \end{cases}$$



$\Delta u = 0$

$u|_{\partial\Omega} = \alpha(x,y)$

Known!



$\Delta u = Q(x,y)$

$u|_{\partial\Omega} = 0.$

Method of eigenfct expansion

$$\begin{cases} \Delta u = Q(x,y) \\ u\big|_{\partial\Omega} = 0. \end{cases}$$

Solve $\begin{cases} \Delta\phi + \lambda\phi = 0 \\ \phi\big|_{\partial\Omega} = 0. \end{cases}$

$\lambda_n, \quad \phi_n.$

$$u = \sum_n a_n \phi_n(x,y)$$

$$Q(x,y) = \sum_n q_n \cdot \phi_n(x,y)$$

$$q_n = \frac{\iint Q \cdot \phi_n}{\iint \phi_n \cdot \phi_n}.$$

$$\Delta u = \sum_n a_n (-\lambda_n)\, \phi_n(x,y)$$

$$\Rightarrow \quad a_n = -\frac{q_n}{\lambda_n}.$$