Poisson equation

$$\begin{cases} \Delta u = f(x,y) \\ u|_{\partial\Omega} = g(x,y) \end{cases}$$

$$\begin{cases} \Delta u = f(x,y) \\ u|_{\partial\Omega} = 0 \end{cases}$$ Method of eigen function expansion

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g(x,y) \end{cases}$$

Example: $$\begin{cases} \Delta u = r^2 \\ u(1,\theta) = \cos 2\theta \end{cases}$$

① $$\begin{cases} \Delta u = r^2 \\ u(1,\theta) = 0 \end{cases}$$ $\Rightarrow$ Assume $u(r,\theta) = u(r)$

$$\frac{1}{r}(ru')' = r^2 \Rightarrow (ru')' = r^3$$

$$(ru') = \frac{1}{4}r^4 + C_1$$

$$u' = \frac{1}{4} r^3 + \frac{C_1}{r}$$

$$u = \frac{1}{16} r^4 + C_1 \log r + C_2 .$$

$u(1) = 0$ and $|u(0)| < +\infty$

imply $C_1 = 0$, $C_2 = -\frac{1}{16}$

$$u(r) = \frac{1}{16} r^4 - \frac{1}{16} .$$

(2) $\begin{cases} \Delta u = 0 \\ u(1, \theta) = \cos 2\theta \end{cases}$

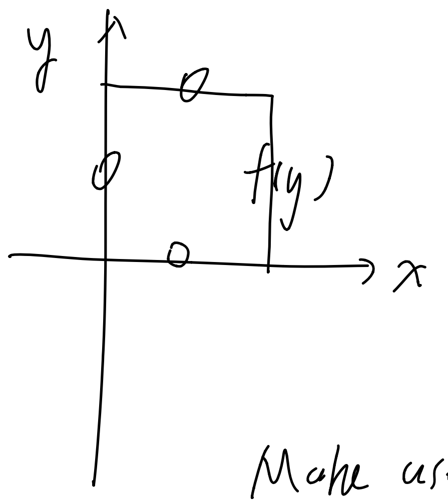
Recall solutions $u(r, \theta) = \sum_{n=0}^{+\infty} A_n r^n \cos n\theta$

$$+ \sum_{n=0}^{+\infty} B_n r^n \sin n\theta$$

so $u(r, \theta) = r^2 \cos 2\theta .$

Solution is $\frac{1}{16} r^4 - \frac{1}{16} + r^2 \cos 2\theta$

Another approach.
to 2D Possion equation.



$$\begin{cases} \Delta u = 0. \\ u(x,0) = u(x,H) = 0 \\ u(0,y) = 0, \quad u(L,y) = f(y) \end{cases}$$

Make use of homogeneous BCs in y-direction.

Write $u(x,y) = \sum_{n=1}^{+\infty} A_n(x) \cdot \sin \frac{n\pi}{H} y$.

$$\Delta u = \sum_n \left(A_n''(x) - \left(\frac{n\pi}{H}\right)^2 A_n(x)\right) \sin \frac{n\pi}{H} y = 0$$

$$\Rightarrow A_n''(x) = \left(\frac{n\pi}{H}\right)^2 A_n(x).$$

Use BCs $\begin{cases} u(0,y) = 0 \\ u(L,y) = f(y) \end{cases}$ to solve

$A_n(x)$.

Fourier Transform see lect 25.