Solve Heat equation or wave equation on infinite domain.

$$u_t = k u_{xx} \qquad -\infty < x < +\infty .$$

$$u(x,0) = f(x).$$

$$\lim_{x \to \pm\infty} u(x,t) \to 0 .$$

Seperation of variables,

$$u(x,t) = \phi(x) \cdot G(t)$$

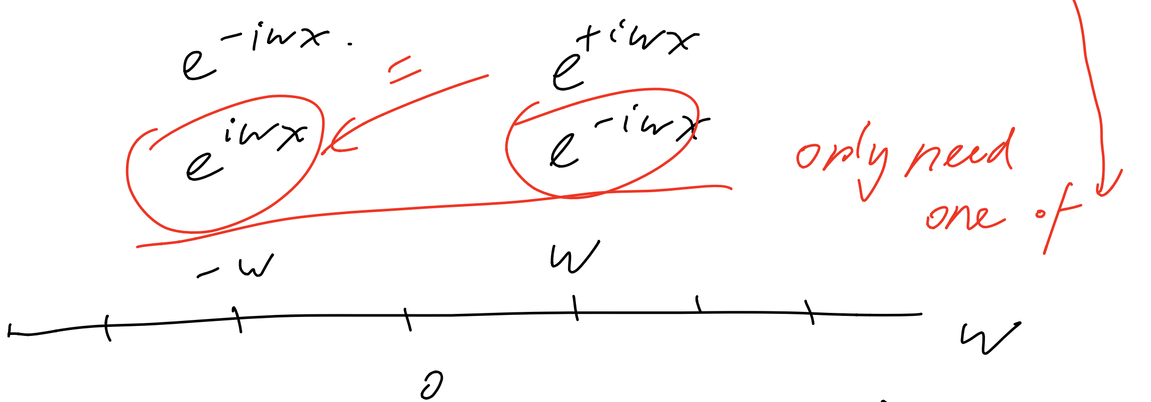
$$\left\{ \begin{array}{l} \phi''(x) + \lambda \phi(x) = 0 \\ G'(t) = -\lambda G(t). \end{array} \right.$$

since $\lim_{x \to \pm\infty} |u(x,t)| < +\infty$.

$\lambda \geq 0$, eigen functions are $\sin\sqrt{\lambda}x$
$\cos\sqrt{\lambda}x$.

① Replace λ by $\lambda = w^2$, $w \geq 0$

② Use complex functions e^{iwx}, e^{-iwx}



③ Resolve the multiplicity of eigenfunctions by allowing negative w.

④ Write solution as integral over the w-parameters. $u(x,t) = \int_{-\infty}^{+\infty} c(w) e^{-iwx} e^{-kw^2 t} \, dw$

$$c(w) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) \cdot e^{iwx} \, dx$$

$$f(x) = \int_{-\infty}^{+\infty} c(w) e^{-iwx} \, dw.$$

Fourier Transform

$$\hat{f}(w) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) \cdot e^{iwx}\, dx$$

Inverse Fourier Transform

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(w) \cdot e^{-iwx}\, dw.$$

Important property:

① linearity :

$$\widehat{f+g}(w) = \hat{f}(w) + \hat{g}(w)$$

② Derivative :

$$\hat{f'}(w) = -iw\, \hat{f}(w)$$

Pf: $$\hat{f'}(w) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f'(x) \cdot e^{iwx}\, dx$$

$$\overset{IBP}{=} \frac{1}{2\pi} \left(- \int_{-\infty}^{+\infty} f(x) \cdot (iw)\, e^{iwx}\, dx \right)$$

$$= -iw \cdot \hat{f}(w).$$

Solve Heat equation by Fourier Transform.

(see lecture 26).